A PSEUDOCODE FOR RS*

Algorithm 3 RS* (Harsha et al., 2007)

```
Require: p, q
n ← 0
q' ← 0                                      ▷ Portion of distribution simulated thus far

repeat
    n ← n + 1
    z ← simulate(n, p)                      ▷ Generate candidate
    p' ← (1 − sum(q')) · p                  ▷ Scaled proposal distribution
    a ← min(1, (q − q')/p')                 ▷ Acceptance probability
    q' ← q' + a · p'
until uniform(n) < a(z)                     ▷ Accept/reject candidate

return n
```

Algorithm 3 (RS*) contains pseudocode for the approach taken by Harsha et al. (2007) to prove a one-shot achievability result for the coding cost of reverse channel coding, the "one-shot reverse Shannon theorem". Our algorithm computes slightly different intermediate results and also differs in notation from the proofs of Harsha et al. (2007). For pseudocode closer in style to the paper, see Appendix A of Havasi et al. (2019).

The algorithm effectively divides a distribution into slices and each iteration attempts to sample from one of these slices. The success probability in each iteration is proportional to the probability mass contained in a slice.

For an intuitive understanding of the algorithm, assume that `p` and `q` are vectors representing categorical proposal and target distributions, respectively. At the beginning of an iteration, `q'` represents the portion of the target distribution which we have already attempted to sample from (a sum of previously considered slices). We always have $\texttt{q'}(z) \leq \texttt{q}(z)$ and `sum(q')` is the probability of having accepted a candidate by the current iteration. $\texttt{p'}(z)$ is thus the probability of reaching the nth iteration *and* producing the candidate z. The vector `a` provides an acceptance probability for each value of the candidate so that $\texttt{a} \cdot \texttt{p'}$ gives the probability of reaching the nth iteration, sampling a candidate value, and accepting it.

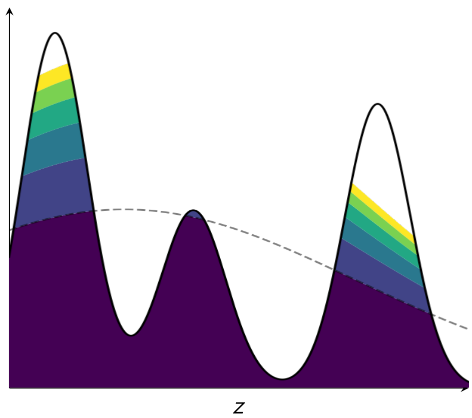

Figure 3: Visualization of 6 iterations of RS*. The solid line corresponds to the target distribution `q` while the dashed line indicates the proposal distribution `p`. Shaded regions correspond to $\texttt{a} \cdot \texttt{p'}$.

The main difference between the algorithm described above and rejection sampling is in the calculation of the acceptance probability. If we used the acceptance probability

$$\texttt{a} \leftarrow \texttt{min}(1, w_{\min} \cdot (\texttt{q} - \texttt{q'})/\texttt{p'}) \tag{1}$$

instead, the algorithm would already reduce to rejection sampling. Note that in this case `min` would not be needed since $w_{\min}$ already makes sure that the acceptance probabilities do not exceed 1. Not using $w_{\min}$ allows RS* to accept candidates with higher probability. The algorithm corrects for this "greedy" approach by targeting $\texttt{q} - \texttt{q'}$ instead of `q` in each iteration. The algorithm is visualized in Figure 3.

Note that RS* requires integration and pointwise multiplication of vectors or functions where other algorithms only require evaluation of densities at a single point. In practice, this means RS* is computationally more demanding and more difficult to implement, especially for continuous distributions.

B PSEUDOCODE FOR ORDERED RANDOM CODING

Algorithm 4 Ordered random coding (ORC)

Require: $\texttt{p}, \texttt{q}, w_{\min}, N$

1: $t, n, s^* \leftarrow 0, 1, \infty$

2: **repeat**

3: $\quad z \leftarrow \texttt{simulate}(n, \texttt{p})$ ▷ Candidate generation

4: $\quad v \leftarrow N/(N - n + 1)$

5: $\quad t \leftarrow t + v \cdot \texttt{expon}(n, 1)$ ▷ Candidate scoring

6: $\quad s \leftarrow t \cdot \texttt{p}(z)/\texttt{q}(z)$

7: $\quad$ **if** $s < s^*$ **then** ▷ Accept/reject candidate

8: $\qquad s^* \leftarrow s$

9: $\qquad n^* \leftarrow n$

10: $\quad$ **end if**

11: $\quad n \leftarrow n + 1$

12: **until** $s^* \leq t \cdot w_{\min}$ **or** $n > N$

13: **return** n^*

Algorithm 4 contains pseudocode for ordered random coding. In contrast to the Poisson functional representation, the algorithm considers only a finite number of candidates N, which means that it can still be used in practice when $w_{\min} = 0$ (either because the density ratio is unbounded or a bound is unknown). Another difference is that the exponential variables are weighted.

We note that for better numerical accuracy, log-density ratios and log-sum-exp operations should be used in practice where the pseudocode uses density ratios and sums (lines 5 and 6).

C PROOF OF THEOREM 1

Theorem 1. *Let* $\mathbf{Z}_n \sim p$ *and* $G_n \sim \text{Gumbel}(0, 1)$ *for* $n \in \{1, \ldots, N\}$ *(i.i.d.). Let* $\tilde{G}_n$ *be the result of sorting the random variables in decreasing order such that* $\tilde{G}_1 \geq \cdots \geq \tilde{G}_N$*. We further define*

$$N^* = \operatorname*{argmax}_{n \leq N} \log \frac{q(\mathbf{Z}_n)}{p(\mathbf{Z}_n)} + G_n, \qquad \tilde{N}^* = \operatorname*{argmax}_{n \leq N} \log \frac{q(\mathbf{Z}_n)}{p(\mathbf{Z}_n)} + \tilde{G}_n. \tag{2}$$

Then $\mathbf{Z}_{N^*} \sim \mathbf{Z}_{\tilde{N}^*}$.

Proof. First, let us define the functions

$$n^*(\mathbf{z}_1, \ldots, \mathbf{z}_N, g_1, \ldots, g_N) = \text{argmax}_n \log \frac{q(\mathbf{z}_n)}{p(\mathbf{z}_n)} + g_n \tag{3}$$

and

$$z(\mathbf{z}_1, \ldots, \mathbf{z}_N, g_1, \ldots, g_N) = \mathbf{z}_{n^*(\mathbf{z}_1, \ldots, \mathbf{z}_N, g_1, \ldots, g_N)}. \tag{4}$$

It is not difficult to see that

$$z(\mathbf{z}_{\sigma(1)}, \ldots, \mathbf{z}_{\sigma(N)}, g_{\sigma(1)}, \ldots, g_{\sigma(N)}) = z(\mathbf{z}_1, \ldots, \mathbf{z}_N, g_1, \ldots, g_N) \tag{5}$$

for any permutation σ of the indices $1, \ldots, N$. Also note that since the candidates are i.i.d. and therefore exchangeable, permuting the candidates does not change their distribution,

$$\mathbf{Z}_1, \ldots, \mathbf{Z}_N \sim \mathbf{Z}_{\sigma(1)}, \ldots, \mathbf{Z}_{\sigma(N)} \sim \prod_n p, \tag{6}$$

and that this is true for any permutation that is independent of $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$ even if the permutation depends on the G_n. We therefore have

$$z(\mathbf{Z}_{\sigma(1)}, \ldots, \mathbf{Z}_{\sigma(N)}, G_1, \ldots, G_N) \sim z(\mathbf{Z}_1, \ldots, \mathbf{Z}_N, G_1, \ldots, G_N). \tag{7}$$

Choose $\tilde{\sigma}$ such that $\tilde{G}_n = G_{\tilde{\sigma}(n)}$. Then

$$\begin{aligned}
\mathbf{Z}_{\tilde{N}^*} &= z(\mathbf{Z}_1, \ldots, \mathbf{Z}_N, G_{\tilde{\sigma}(1)}, \ldots, G_{\tilde{\sigma}(N)}) && (8)\\
&= z(\mathbf{Z}_{\tilde{\sigma}^{-1}(1)}, \ldots, \mathbf{Z}_{\tilde{\sigma}^{-1}(N)}, G_1, \ldots, G_N) && (9)\\
&\sim z(\mathbf{Z}_1, \ldots, \mathbf{Z}_N, G_1, \ldots, G_N) && (10)\\
&= \mathbf{Z}_{N^*}. && (11)
\end{aligned}$$

□

D PROOF OF THEOREM 2

We first prove the following more precise result which follows from the results of Havasi et al. (2019) and Chatterjee and Diaconis (2018).

Lemma 1. *Let* $\tilde{q}$ *be the distribution of* $\mathbf{Z}_{\tilde{N}^*}$*, where* $\tilde{N}^*$ *is defined as in Theorem 1. If the number of candidates is* $N = 2^{D_{\mathrm{KL}}[q \,\|\, p] + t}$ *for some* $t \geq 0$ *and* $\mathbf{Z} \sim q$*, then*

$$D_{\mathrm{TV}}[\tilde{q}, q] \leq 4\epsilon \tag{12}$$

where

$$\epsilon = \left(2^{-t/4} + 2\sqrt{\mathbb{P}\left(\log \frac{q(\mathbf{Z})}{p(\mathbf{Z})} > D_{\mathrm{KL}}[q \,\|\, p] + t/2\right)}\right)^{\frac{1}{2}}. \tag{13}$$

Proof. If $\varepsilon \geq 1/4$ then Eq. 12 is automatically true and there is nothing left to show. Assume therefore that q, p, and t are such that $\varepsilon < 1/4$.

Since $\mathbf{Z}_{\tilde{N}^*} \sim \mathbf{Z}_{N^*}$ by Theorem 1, $\tilde{q}$ is also the distribution of $\mathbf{Z}_{N^*}$. Let $\Omega = \{\mathbf{Z}_1, \ldots, \mathbf{Z}_N\}$ be the set of candidates and let $\tilde{q}_\Omega$ be the distribution of $\mathbf{Z}_{N^*}$ for a fixed set of candidates. That is,

$$\tilde{q}_\Omega(\mathbf{z}) = \sum_{n=1}^{N} \pi(n)\delta(\mathbf{z} - \mathbf{Z}_n) \tag{14}$$

where $\pi(n) \propto q(\mathbf{Z}_n)/p(\mathbf{Z}_n)$. Theorem 3.2 of Havasi et al. (2019) tells us that

$$\mathbb{P}\left(\left|\mathbb{E}_{q_\Omega}[f(\tilde{\mathbf{Z}})] - \mathbb{E}_q[f(\mathbf{Z})]\right| \geq 2\|f\|_q \frac{\epsilon}{1-\epsilon}\right) < 2\epsilon, \tag{15}$$

for any measurable function f, where $\|f\|_q = \sqrt{\mathbb{E}_q[f(\mathbf{Z})^2]}$ and the probability arises due to the random set of candidates Ω. We choose

$$f(\mathbf{z}) = \begin{cases} 1 & \text{if } \tilde{q}(\mathbf{z}) > q(\mathbf{z}), \\ -1 & \text{else.} \end{cases} \tag{16}$$

We further define the event

$$A = \left[\!\!\left[\left|\mathbb{E}_{q_\Omega}[f(\tilde{\mathbf{Z}})] - \mathbb{E}_q[f(\mathbf{Z})]\right| \geq \|f\|_q \frac{2\epsilon}{1-\epsilon} \right]\!\!\right], \tag{17}$$

where $[\![\cdot]\!]$ is 1 if its argument is true and 0 otherwise. We have

$$\begin{aligned}
2D_{\mathrm{TV}}[\tilde{q}, q] &= |\mathbb{E}_{\tilde{q}}[f(\tilde{\mathbf{Z}})] - \mathbb{E}_q[f(\mathbf{Z})]| && (18)\\
&= |\mathbb{E}_\Omega[\mathbb{E}_{\tilde{q}_\Omega}[f(\tilde{\mathbf{Z}})]] - \mathbb{E}_q[f(\mathbf{Z})]| && (19)\\
&\leq \mathbb{E}_\Omega[|\mathbb{E}_{\tilde{q}_\Omega}[f(\tilde{\mathbf{Z}})] - \mathbb{E}_q[f(\mathbf{Z})]|] && (20)\\
&= P(A=1)\mathbb{E}_\Omega[|\mathbb{E}_{\tilde{q}_\Omega}[f(\tilde{\mathbf{Z}})] - \mathbb{E}_q[f(\mathbf{Z})]| \mid A=1] + P(A=0)\mathbb{E}_\Omega[|\mathbb{E}_{\tilde{q}_\Omega}[f(\tilde{\mathbf{Z}})] - \mathbb{E}_q[f(\mathbf{Z})]| \mid A=0] && (21)\\
&\leq 2P(A=1) + (1 - P(A=1))\tfrac{2\epsilon}{1-\epsilon}\|f\|_q && (22)\\
&= 2P(A=1)\left(1 - \tfrac{\epsilon}{1-\epsilon}\right) + \tfrac{2\epsilon}{1-\epsilon} && (23)\\
&\leq 4\epsilon\left(1 - \tfrac{\epsilon}{1-\epsilon}\right) + \tfrac{2\epsilon}{1-\epsilon} && (24)\\
&\leq 4\epsilon + 4\epsilon && (25)\\
&= 8\epsilon, && (26)
\end{aligned}$$

where Eq. 18 is a known identity (e.g., Sriperumbudur et al., 2009), Eq. 19 follows from Jensen's inequality, Eq. 22 follows from the definitions of f and A, Eq. 23 follows from $\|f(\mathbf{z})\| = 1$, and Eq. 24 follows from Eq. 15. □

Theorem 2. *Let $\tilde{q}$ be the distribution of $\mathbf{Z}_{\tilde{N}^*}$, where $\tilde{N}^*$ is defined as in Theorem 1. If the number of candidates is $N = 2^{D_{\mathrm{KL}}[q \,\|\, p]+t}$ and $p(\mathbf{z})/q(\mathbf{z}) \geq w_{\min} > 0$ for all $\mathbf{z}$, then*

$$D_{\mathrm{TV}}[\tilde{q}, q] = O(2^{-t/8}) \tag{27}$$

Proof. By Lemma 1, we have

$$D_{\mathrm{TV}}[\tilde{q}, q] \leq 4\epsilon \tag{28}$$

where

$$\epsilon = \left(2^{-t/4} + 2\sqrt{\mathbb{P}\left(\log\frac{q(\mathbf{Z})}{p(\mathbf{Z})} > D_{\mathrm{KL}}[q \parallel p] + t/2\right)}\right)^{\frac{1}{2}}. \tag{29}$$

To prove our claim we need to bound ϵ. Define

$$l(\mathbf{z}) = \max(0, \log q(\mathbf{z}) - \log p(\mathbf{z})). \tag{30}$$

By Claim A.2 of Harsha et al. (2007), we have

$$\mathbb{E}_q[l(\mathbf{Z})] = \mathbb{E}_q\left[\max\left(0, \log\frac{q(\mathbf{Z})}{p(\mathbf{Z})}\right)\right] = \mathbb{E}_q\left[\log\frac{q(\mathbf{Z})}{p(\mathbf{Z})} - \min\left(0, \log\frac{q(\mathbf{Z})}{p(\mathbf{Z})}\right)\right] \leq D_{\mathrm{KL}}[q \parallel p] + e^{-1}\log e. \tag{31}$$

Let $B = -\log w_{\min}$ so that $l(\mathbf{z}) \leq B$ for all $\mathbf{z}$. For sufficiently large t, we have

$$\begin{aligned}
\mathbb{P}\left(\log\frac{q(\mathbf{Z})}{p(\mathbf{Z})} > D_{\mathrm{KL}}[q \parallel p] + t/2\right) &= \mathbb{P}\left(l(\mathbf{Z}) > D_{\mathrm{KL}}[q \parallel p] + t/2\right) && (32)\\
&= \mathbb{P}\left(l(\mathbf{Z}) > \mathbb{E}_q[l(\mathbf{Z})] + D_{\mathrm{KL}}[q \parallel p] - \mathbb{E}_q[l(\mathbf{Z})] + t/2\right) && (33)\\
&\leq \exp\left(-(D_{\mathrm{KL}}[q \parallel p] - \mathbb{E}_q[l(\mathbf{Z})] + t/2)^2/B^2\right) && (34)\\
&\leq \exp\left(-(t/2 - e^{-1}\log e)^2/B^2\right) && (35)
\end{aligned}$$

where the first equality follows from the non-negativity of the KL divergence, Eq. 34 follows from Hoeffding's inequality, and the last inequality follows from Eq. 31 and assuming $t/2 \geq e^{-1}\log e$. Thus, for large enough t we have

$$\begin{aligned}
\epsilon &\leq \left(2^{-t/4} + 2\exp\left(-\frac{1}{2B^2}(t/2 - e^{-1}\log e)^2\right)\right)^{\frac{1}{2}} && (36)\\
&\leq 2^{-t/8} + \sqrt{2}\exp\left(-\frac{1}{4B^2}(t/2 - e^{-1}\log e)^2\right) && (37)\\
&= O(2^{-t/8}), && (38)
\end{aligned}$$

where the second inequality follows from $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, and therefore

$$D_{\mathrm{TV}}[\tilde{q}, q] = O(2^{-t/8}). \tag{39}$$

□

E PROOF OT THEOREM 3

Theorem 3. *Let S_n be exponentially distributed RVs and $\mathbf{Z}_n \sim p$ for all $n \in \mathbb{N}$ (i.i.d.), and let*

$$T_n = \sum_{m=1}^{n} S_m, \qquad \tilde{T}_{N,n} = \sum_{m=1}^{n} \frac{N}{N-m+1} S_m \tag{40}$$

for $N \in \mathbb{N}$. Further, let

$$N^*_{\mathrm{PFR}} = \operatorname*{argmin}_{n \in \mathbb{N}} T_n \frac{p(\mathbf{Z}_n)}{q(\mathbf{Z}_n)}, \qquad N^*_{\mathrm{ORC}} = \operatorname*{argmin}_{n \leq N} \tilde{T}_{N,n} \frac{p(\mathbf{Z}_n)}{q(\mathbf{Z}_n)}. \tag{41}$$

Then $N^*_{\mathrm{ORC}} \leq N^*_{\mathrm{PFR}}$. Further, there exists an $M \in \mathbb{N}$ such that for all $N \geq M$ we have $N^*_{\mathrm{ORC}} = N^*_{\mathrm{PFR}}$.

Proof. We first show that $N^*_{\text{ORC}} \leq N^*_{\text{PFR}}$. For any $\Delta \geq 0$ and n with $n + \Delta \leq N$ we have

$$\frac{\tilde{T}_{N,n+\Delta}}{\tilde{T}_{N,n}} = \frac{\sum_{m=1}^{n+\Delta} \frac{N}{N-m+1} S_m}{\sum_{m=1}^{n} \frac{N}{N-m+1} S_m} \tag{42}$$

$$= \frac{\sum_{m=1}^{n+\Delta} \frac{N-n+1}{N-m+1} S_m}{\sum_{m=1}^{n} \frac{N-n+1}{N-m+1} S_m} \tag{43}$$

$$= \frac{\sum_{m=1}^{n} \frac{N-n+1}{N-m+1} S_m + \sum_{m=n+1}^{n+\Delta} \frac{N-n+1}{N-m+1} S_m}{\sum_{m=1}^{n} \frac{N-n+1}{N-m+1} S_m} \tag{44}$$

$$\geq \frac{\sum_{m=1}^{n} \frac{N-n+1}{N-m+1} S_m + \sum_{m=n+1}^{n+\Delta} S_m}{\sum_{m=1}^{n} \frac{N-n+1}{N-m+1} S_m} \tag{45}$$

$$\geq \frac{\sum_{m=1}^{n} S_m + \sum_{m=n+1}^{n+\Delta} S_m}{\sum_{m=1}^{n} S_m} \tag{46}$$

$$= \frac{T_{n+\Delta}}{T_n} \tag{47}$$

That is, $\tilde{T}_{N,n}$ increases at a faster rate than T_n. Eq. 45 is true because $S_m \geq 0$ and $(N - n + 1)/(N - m + 1) > 1$ for $m \geq n + 1$. Eq. 46 is true because $(N - n + 1)/(N - m + 1) \leq 1$ for $m \leq n$ and

$$f : [0, \infty) \to [0, \infty), \quad t \mapsto \frac{t + a}{t} \tag{48}$$

is a monotonically decreasing function for any $a > 0$.

Now assume that N^*_{PFR} takes on some value n^*. If $n^* \geq N$ then $N^*_{\text{ORC}} \leq N^*_{\text{PFR}}$ since $N^*_{\text{ORC}} \leq N$. If $n^* < N$, to show that N^*_{ORC} does not exceed N^*_{PFR} it is enough to show that[1]

$$\tilde{T}_{N,n^*} \frac{p(\mathbf{Z}_{n^*})}{q(\mathbf{Z}_{n^*})} \leq \tilde{T}_{N,n^*+\Delta} \frac{p(\mathbf{Z}_{n^*+\Delta})}{q(\mathbf{Z}_{n^*+\Delta})} \tag{49}$$

for any $\Delta > 0$ with $n^* + \Delta \leq N$, since N^*_{ORC} then must be either n^* or take on a smaller value. By definition of N^*_{PFR}, we have

$$T_{n^*} \frac{p(\mathbf{Z}_{n^*})}{q(\mathbf{Z}_{n^*})} \leq T_{n^*+\Delta} \frac{p(\mathbf{Z}_{n^*+\Delta})}{q(\mathbf{Z}_{n^*+\Delta})} \tag{50}$$

for any $\Delta > 0$. From this and Eqs. 42 to 47 it follows that

$$\frac{p(\mathbf{Z}_{n^*})}{q(\mathbf{Z}_{n^*})} \leq \frac{T_{n^*+\Delta}}{T_{n^*}} \frac{p(\mathbf{Z}_{n^*+\Delta})}{q(\mathbf{Z}_{n^*+\Delta})}, \tag{51}$$

$$\frac{p(\mathbf{Z}_{n^*})}{q(\mathbf{Z}_{n^*})} \leq \frac{\tilde{T}_{N,n^*+\Delta}}{\tilde{T}_{N,n^*}} \frac{p(\mathbf{Z}_{n^*+\Delta})}{q(\mathbf{Z}_{n^*+\Delta})}, \tag{52}$$

$$\tilde{T}_{N,n} \frac{p(\mathbf{Z}_{n^*})}{q(\mathbf{Z}_{n^*})} \leq \tilde{T}_{N,n^*+\Delta} \frac{p(\mathbf{Z}_{n^*+\Delta})}{q(\mathbf{Z}_{n^*+\Delta})}, \tag{53}$$

$$\tag{54}$$

which concludes the proof of $N^*_{\text{ORC}} \leq N^*_{\text{PFR}}$.

[1] We assume that in case of a tie, argmin returns the smaller index.

Next, we show that $N^*_{\text{ORC}} = N^*_{\text{PFR}}$ for large enough N. First note that we can equivalently define N^*_{ORC} as follows,

$$N' = \min\{N^*_{\text{PFR}}, N\}, \qquad N^*_{\text{ORC}} = \operatorname*{argmin}_{n \leq N'} \tilde{T}_{N,n} \frac{p(\mathbf{Z}_n)}{q(\mathbf{Z}_n)}. \tag{55}$$

We have

$$\lim_{N\to\infty} \tilde{T}_{N,n} = \sum_{m=1}^{n} \left(\lim_{N\to\infty} \frac{N}{N-m+1} \right) S_m = T_n \tag{56}$$

for all $n \leq N^*_{\text{PFR}}$ and therefore

$$\lim_{N\to\infty} N^*_{\text{ORC}} = \operatorname*{argmin}_{n \leq N^*_{\text{PFR}}} \lim_{N\to\infty} \tilde{T}_{N,n} \frac{p(\mathbf{Z}_n)}{q(\mathbf{Z}_n)} = \operatorname*{argmin}_{n \leq N^*_{\text{PFR}}} T_n \frac{p(\mathbf{Z}_n)}{q(\mathbf{Z}_n)} = N^*_{\text{PFR}}. \tag{57}$$

□

F PROOF OF COROLLARY 1

Corollary 1. *Let $C = \mathbb{E}_{\mathbf{X}}[D_{\text{KL}}[q_{\mathbf{X}} \parallel p]]$ and let N^*_{ORC} be defined as in Theorem 3. Then*

$$H[N^*_{\text{ORC}}] < C + \log(C+1) + 4. \tag{58}$$

Proof. Let N^*_{PFR} be defined as in Theorem 3. Li and El Gamal (2018, Appendix A) showed that

$$\mathbb{E}[\log N^*_{\text{PFR}} \mid \mathbf{X} = \mathbf{x}] \leq D_{\text{KL}}[q_{\mathbf{x}} \parallel p] + e^{-1} \log e + 1. \tag{59}$$

While Li and El Gamal (2018) were only considering the case where $p(\mathbf{z}) = \mathbb{E}_{\mathbf{X}}[q_{\mathbf{X}}(\mathbf{z})]$, their proof of the above statement does not make use of this assumption. Since $N^*_{\text{ORC}} \leq N^*_{\text{PFR}}$, we also have

$$\mathbb{E}[\log N^*_{\text{ORC}}] \leq \mathbb{E}[\log N^*_{\text{PFR}}] \leq \mathbb{E}_{\mathbf{X}}[D_{\text{KL}}[q_{\mathbf{X}} \parallel p]] + e^{-1} \log e + 1. \tag{60}$$

Li and El Gamal (2018, Appendix B) further showed that for any random variable N^* with values in $\mathbb{N}$, we have

$$\mathbb{E}[-\log p_\lambda(N^*)] \leq \mathbb{E}[\log N^*] + \log(\mathbb{E}[\log N^*] + 1) + 1, \tag{61}$$

where $p_\lambda(n) \propto n^{-\lambda}$ is a Zipf distribution with

$$\lambda = 1 + 1/\mathbb{E}[\log N^*]. \tag{62}$$

Applied to N^*_{ORC} and using Eq. 60, we get

$$\begin{aligned}
H[N^*_{\text{ORC}}] &\leq \mathbb{E}[-\log p_\lambda(N^*_{\text{ORC}})] && (63)\\
&\leq \mathbb{E}[\log N^*_{\text{ORC}}] + \log(\mathbb{E}[\log N^*_{\text{ORC}}] + 1) + 1 && (64)\\
&\leq C + \log(C + e^{-1} \log e + 2) + e^{-1} \log e + 2 && (65)\\
&< C + \log(C+1) + 4. && (66)
\end{aligned}$$

□

G PROOF OF THEOREM 4

Let us first briefly repeat the relevant definitions from the main text. We have

$$r_{\mathbf{x}}(\mathbf{z}) = \begin{cases} 1 & \text{if } \mathbf{z} \in \mathbf{c}_{\mathbf{x}} + [-0.5, 0.5)^D, \\ 0 & \text{else,} \end{cases} \tag{67}$$

where $\mathbf{c}_{\mathbf{x}}$ is chosen such that the support of $r_{\mathbf{x}}$ is contained within $[0, M_1) \times \cdots [0, M_D)$. Candidates are generated via dithered quantization

$$\mathbf{U}_n \sim \text{Uniform}([0,1)^D), \qquad \mathbf{K}_n = \lfloor \mathbf{c}_{\mathbf{x}} - \mathbf{U}_n \rceil, \qquad \mathbf{Z}_n = \mathbf{K}_n + \mathbf{U}_n, \tag{68}$$

so that $\mathbf{Z}_n \sim r_{\mathbf{x}}$. One of the candidates is then selected according to

$$\tilde{T}_{N,n} = \sum_{m=1}^{n} \frac{N}{N-m+1} S_m, \qquad N^* = \operatorname*{argmin}_{n \leq N} \tilde{T}_{N,n} \frac{r_{\mathbf{x}}(\mathbf{Z}_n)}{q_{\mathbf{x}}(\mathbf{Z}_n)}, \tag{69}$$

where the support of $q_{\mathbf{x}}$ is assumed to be contained in the support of $r_{\mathbf{x}}$. For notational convenience, further define

$$\mathbf{K}^* = \mathbf{K}_{N^*}, \qquad \mathbf{U}^* = \mathbf{U}_{N^*}, \qquad \mathbf{Z}^* = \mathbf{Z}_{N^*}. \tag{70}$$

Note that $\mathbf{Z}^* = \mathbf{K}^* + \mathbf{U}^*$. The following theorem bounds the coding cost of optimally encoding N^* and $\mathbf{K}^*$.

Theorem 4. *Let N^* and $\mathbf{K}^*$ be defined as in Eqs. 69 and 70 and let p be the uniform distribution over $[0, M_1) \times \cdots \times [0, M_D)$ for some $M_i \in \mathbb{N}$. Then*

$$H[N^*, \mathbf{K}^*] < C + \log(C - \textstyle\sum_i \log M_i + 1) + 4,$$

where $C = \mathbb{E}_{\mathbf{X}}[D_{\text{KL}}[q_{\mathbf{X}} \parallel p]]$.

Proof. By Corollary 1, we have

$$H[N^*] < C' + \log(C' + 1) + 4 \tag{71}$$

where $C' = \mathbb{E}_{\mathbf{X}}[D_{\text{KL}}[q_{\mathbf{X}} \parallel r_{\mathbf{X}}]] = C - \sum_i \log M_i$, assuming the support of $q_{\mathbf{x}}$ is contained in the support of $r_{\mathbf{x}}$.

Next consider the coding cost of $\mathbf{K}^*$. Note that for each entry K_i^* in $\mathbf{K}^*$ we must have $0 \leq K_i^* < M_i$ since otherwise $K_i^* + U_i^* < 0$ or $K_i^* + U_i^* \geq M_i$, that is, $\mathbf{Z}^* = \mathbf{K}^* + \mathbf{U}^*$ would be outside the support of $r_{\mathbf{X}}$. Hence,

$$H[\mathbf{K}^*] \leq \sum_i \log M_i \tag{72}$$

$$= \log \frac{1}{p(\mathbf{z})} \tag{73}$$

$$= \mathbb{E}_{\mathbf{X}}\left[\mathbb{E}_{\mathbf{Z} \sim q_{\mathbf{X}}}\left[\log \frac{r_{\mathbf{X}}(\mathbf{Z})}{p(\mathbf{Z})}\right]\right] \tag{74}$$

$$= \mathbb{E}_{\mathbf{X}}\left[\mathbb{E}_{\mathbf{Z} \sim q_{\mathbf{X}}}\left[\log \frac{q_{\mathbf{X}}(\mathbf{Z}) r_{\mathbf{X}}(\mathbf{Z})}{p(\mathbf{Z}) q_{\mathbf{X}}(\mathbf{Z})}\right]\right] \tag{75}$$

$$= C - C'. \tag{76}$$

Taken together, we have

$$H[N^*, \mathbf{K}^*] \leq H[N^*] + H[\mathbf{K}^*] < C' + \log(C' + 1) + 4 + C - C' = C + \log(C - \sum_i \log M_i + 1) + 4, \tag{77}$$

proving the claim. □

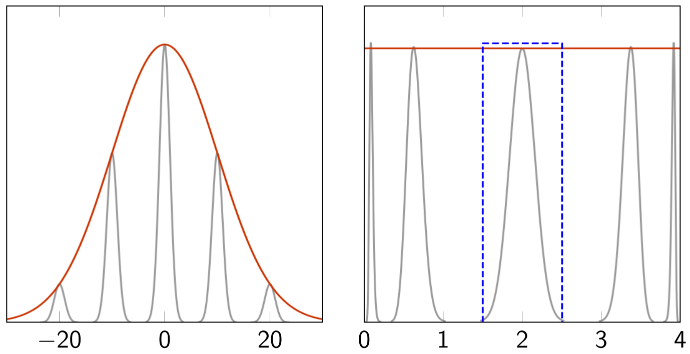

Figure 4: Target and candidate generating distributions before and after transformation. *Left:* Gray curves indicate target distributions $\tilde{q}_{\mathbf{x}}$ (scaled for visualization purposes) while the red curve indicates the marginal distribution of $\tilde{\mathbf{Z}}$, that is, $\mathbb{E}[\tilde{q}_{\mathbf{x}}]$. *Right:* The same distributions after transformation with Φ_{σ^2+1} and scaling by $M = 4$. The dashed blue line indicates $r_{\mathbf{x}}$, which has to be wide enough to cover the support of the widest transformed distribution.

H HYBRID CODING FOR GAUSSIAN DISTRIBUTIONS

Let $\tilde{q}_{\mathbf{x}}$ be a truncated Gaussian with mean $\mathbf{x}$ and covariance $\mathbf{I}$ and let θ be the fraction of mass which has been truncated. We assume that the mean is itself Gaussian distributed with covariance $\sigma^2\mathbf{I}$ so that a Gaussian with covariance $(\sigma^2+1)\mathbf{I}$ is a suitable candidate generating distribution $\tilde{p}$.

To make the marginal distribution uniform, we first transform each coordinate with the CDF of a univariate Gaussian with variance σ^2+1, Φ_{σ^2+1}. After this transformation, different target distributions have supports of varying widths. The distribution with the widest support is centered at zero. The support of the truncated Gaussian is limited to the left and right by

$$a = \Phi^{-1}(\theta'/2), \qquad b = \Phi^{-1}(1-\theta'/2), \tag{78}$$

along each coordinate, where $\theta' = 1-(1-\theta)^{1/D}$ and Φ is the CDF of a standard normal. After the transformation, the limits of the support become $\Phi_{\sigma^2+1}(a)$ and $\Phi_{\sigma^2+1}(b)$, respectively, so that we can scale the distributions by

$$M = \left\lfloor \frac{1}{\Phi_{\sigma^2+1}(a) - \Phi_{\sigma^2+1}(b)} \right\rfloor \tag{79}$$

along the ith coordinate while still ensuring that the distributions fit into a unit interval. For $D = 1$, the target distribution becomes

$$q_x(z) = \frac{\tilde{q}_x(\tilde{z})}{M\Phi'_{\sigma^2+1}(\tilde{z})} = \frac{1}{M}\frac{\tilde{q}_x(\tilde{z})}{\mathcal{N}(\tilde{z};0,\sigma^2+1)} \tag{80}$$

where $\tilde{z} = \Phi^{-1}_{\sigma^2+1}(z/M)$ and $z \in [0, M)$. This is visualized in Figure 4. For $D > 1$, we have

$$q_{\mathbf{x}}(\mathbf{z}) = \prod_i q_{x_i}(z_i). \tag{81}$$

For $w_{\min}$, we choose

$$\inf_{\mathbf{z}} \frac{p(\mathbf{z})}{q_{\mathbf{x}}(\mathbf{z})} = \inf_{\tilde{\mathbf{z}}} \frac{\tilde{p}(\tilde{\mathbf{z}})}{\tilde{q}_{\mathbf{x}}(\tilde{\mathbf{z}})} \geq \inf_{\tilde{\mathbf{z}}} \frac{(1-\theta)\mathcal{N}(\tilde{\mathbf{z}};0,(\sigma^2+1)\mathbf{I})}{\mathcal{N}(\tilde{\mathbf{z}};\mathbf{x},\mathbf{I})} = (1-\theta)\frac{\mathcal{N}(\tilde{\mathbf{z}}_{\min};0,(\sigma^2+1)\mathbf{I})}{\mathcal{N}(\tilde{\mathbf{z}}_{\min};\mathbf{x},\mathbf{I})} = w_{\min} \tag{82}$$

where

$$\tilde{\mathbf{z}}_{\min} = \frac{\sigma^2 + 1}{\sigma^2}\mathbf{x}. \tag{83}$$

is the minimizer of the infimum on the right-hand side. Since the density ratios are invariant under transformation, we can use the same $w_{\min}$ for ORC/PFR and the hybrid coding scheme.

I ADDITIONAL FIGURES

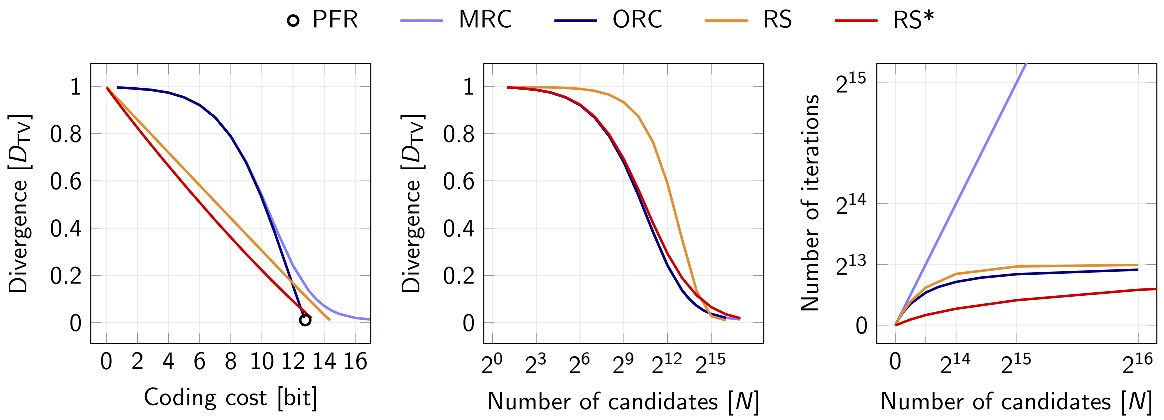

Figure 3: Additional figures for the example of communicating categorical samples. *Left:* The sample quality as a function of the coding cost, as in the main text but for a wider range of values. Note that samples of low quality (high D_{TV}) are rarely of interest. *Middle:* The sample quality as a function of the maximum number of candidates available to an algorithm. *Right:* The average number of candidates considered (that is, the number of iterations before termination) as a function of the maximum number of candidates.

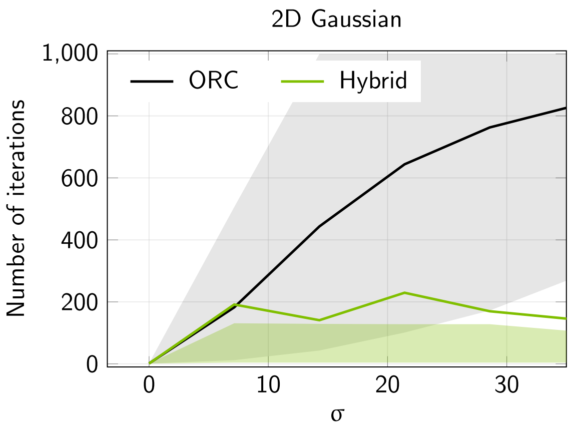

Figure 4: Additional results comparing the hybrid coding scheme with ordered random coding on a 2-dimensional multivariate Gaussian distribution. Here, the number of candidates of ORC was limited to $N = 1000$ but was unlimited for the hybrid coding scheme. Nevertheless, the hybrid coding scheme converges much faster. The average number of iterations of the hybrid coding scheme exceeds the 90th percentile due to outliers.

References

S. Chatterjee and P. Diaconis. The sample size required in importance sampling. *The Annals of Applied Probability*, 28(2):1099–1135, 2018.

P. Harsha, R. Jain, D. McAllester, and J. Radhakrishnan. The Communication Complexity of Correlation. In *Twenty-Second Annual IEEE Conference on Computational Complexity*, pages 10–23, 2007.

M. Havasi, R. Peharz, and J. M. Hernández-Lobato. Minimal Random Code Learning: Getting Bits Back from Compressed Model Parameters. In *International Conference on Learning Representations*, 2019.

C. T. Li and A. El Gamal. Strong Functional Representation Lemma and Applications to Coding Theorems. *IEEE Transactions on Information Theory*, 64(11):6967–6978, 2018. doi: 10.1109/TIT.2018.2865570.

B. K. Sriperumbudur, K. Fukumizu, A. Gretton, B. Schölkopf, and G. R. G. Lanckriet. On integral probability metrics, ϕ-divergences and binary classification, 2009.