# Relativistic Gravitational Atoms

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**ABSTRACT**

Schrödinger's equation has beeen used to investigate theoretically the bound states in a system consisting of two object matter waves held together by their mutual gravitational attraction: a Gravitational Atom. In seeking the stationary stable states of this system we include the normal gravitational attraction and also allow for the gravitational effects of the relativistic increase in mass of the rapidly orbiting objects.

In solving the equations, we find a series of solutions. These are of states corresponding to stationary matter waves that one might infer are stable, and would emit no gravitational radiation except in transitions to other states. Gravitational Atoms in these states may exist and be observable.

From the infinitude of solutions, certain restrictions may be imposed to find physical possible solutions. Only systems composed of objects with an average rest mass less than $10^{-9}$ kg will have non-overlapping Schwartzchild Radii in the Ground State. Restrictions on the rest masses of composing objects are also imposed by requiring energy levels to be less than the estimated mass energy of the observable universe.

Also: some ranges of rest masses of composing objects have states with negative energy levels greater in magnitude than the total rest mass energy of the composing objects. Systems in these states will have a net negative total energy, and so may display a negative mass, and repel normal matter gravitationally.

**INTRODUCTION**

In the study reported here, we consider theoretically the bound states of a gravitational atom consisting of the matter waves of two objects, one light and one heavy, moving only under the influence of their mutual gravitation. The basic qualitative idea is that the gravitational mass of the rapidly orbiting lighter object is increased as suggested in Special Relativity by its orbital kinetic energy. Using this simple model of Special Relativity [eg. Schwartz, 2007], the mass of the lighter object can be written:

$$u = m + T/c^2 = m.[\ 1 + T/(mc^2)\ ] \qquad (6)$$

where in MKS units [Allen, 1964] :

m = rest mass of the light object, kg,
u = travelling mass of the light object, kg,
T = kinetic energy, J,
c = velocity of light = $3.0 \times 10^8$ m/s,
V = potential energy, J,
G = universal constant of gravitation = $6.7 \times 10^{-11}$ $m^3kg^{-1}s^{-2}$,
r = separation of the two objects, m,

$M$ = rest mass of the heavy object, kg,
$h$ = reduced Planck's constant = $1.1 \times 10^{-34}$ J.s

## MATHEMATICAL DETAILS

Quantitatively we may set up the Schrödinger equation to find the characteristic energies of the stationary states of this Gravitational Atom. In this simple exploratory study, we take the approximations that the objects are point masses, and that the lighter object moves around a stationary heavier object centred on the coordinate origin, then:

$$V \approx - GMu/r = - (GMm/r).[\ 1 + T/(mc^2)\ ] \qquad (7)$$

Then with E = total energy of the atom, we have

$$E = T + V \qquad (8)$$

or

$$E = T - (GMm/r).[\ 1 + T/(mc^2)\ ] = T.[1 - GM/(rc^2)] - (GMm/r) \qquad (9)$$

or

$$T.[1 - GM/(rc^2)] - (GMm/r) - E = 0 \qquad (10)$$

Following the usual development to find the stationary states (e.g [Houston, 1959]) we transform this using de Broglie's relationship:

$$T \rightarrow -(h^2/2m).\Delta \qquad (11)$$

where

$h$ = reduced Planck's constant = $1.1 \times 10^{-34}$ J.s
$\Delta$ is the Laplacian operator

Thus we obtain:

$$\{\ (h^2/2m).[1 - GM/(rc^2)]\Delta + GMm/r + E\ \}.\psi = 0 \qquad (12)$$

where

$\psi$ = wave function of the smaller object

Thus

$$\{\ \Delta + 2m.(GMm + E.r)/[\ h^2.(r- GM/c^2)\ ]\ \}.\psi = 0 \qquad (13)$$

Writing this as

$$\{\ \Delta + [\ B + C.r\ ]/[\ r-A\ ]\ \}.\psi = 0 \qquad (14)$$

with

$A = GM/c^2$ (the Zero-Energy Radius of the heavier object) $\qquad (15)$
$B = 2GMm^2 / h^2 \qquad (16)$
$C = 2mE/h^2 \qquad (17)$

Writing:

$$[\ B + C.r\ ]/[\ r-A\ ] = F/[\ r-A\ ] + C.[\ r-A\ ]/[\ r-A\ ] \qquad (18)$$

we have

$$F = AC + B \qquad (19)$$

Changing the dependent variable to s:

$$s = r - A \qquad (20)$$

we obtain the usual form of the hydrogen atom energy equation:

$$\{ \Delta + F/s + C \}.\psi = 0 \qquad (21)$$

Writing $\psi$ in spherical coordinates as the product of a radial function R and an angular function W:

$$\psi = R(s).W(\theta,\varphi) \qquad (22)$$

Assuming an angular momentum k about the point s = 0 (i.e. r = A), we can remove the angular dependence to obtain:

$$(d^2R/ds^2) + (2/s).(dR/ds) + [\ C + F/s - k(k+1)/s^2\ ].R = 0 \qquad (23)$$

with

$$k = 1,2,3,...$$

We seek solutions for R that decay exponentially to zero as $s \rightarrow \infty$ , so let:

$$R = Q(s).e^{-s.v} \qquad (24)$$

where:

$$v^2 = -C = -2mE/h^2 \qquad (25)$$

Then:

$$(d^2Q/ds^2) - 2(v - 1.s).(dQ/ds) + [\ (F-2v)/s - k(k+1)/s^2\ ].Q = 0 \qquad (26)$$

This equation is known to have two solutions, which at the origin behave as

$$Q \approx s^k \quad \text{or} \quad s^{-(k+1)} \qquad (27)$$

The normalisation condition requires a finite value of

$$\int_0^{\infty} \int_0^{\pi} \int_0^{2\pi} \psi.\psi^*.r^2.\sin(\theta).d\varphi.d\theta.dr \qquad (28)$$

or a finite value of

$$\int_{-A}^{\infty} \int_0^{\pi} \int_0^{2\pi} \psi.\psi^*.(s+A)^2.\sin(\theta).d\varphi.d\theta.ds \qquad (29)$$

which requires a finite value of

$$\int_{-A}^{\infty} Q^2.e^{(-2s.v)}.(s+A)^2.ds \qquad (30)$$

In order to examine the solutions near the origin of s: let P(s) be a polynomial in s.

$$Q = s^k.P(s) \qquad (31)$$

where P(s) is a polynomial in s. The solutions of the form $s^k$ are those of conventionally assigned to the hydrogen atom, but the solutions of the form $s^{-(k+1)}$ are usually rejected because the singularity at r = 0 impedes normalisation of ψ. In the current case, the singularity is not at the lower limit of the integral, and these solutions may have some physical meaning, even though the integral encompasses the singularity.

Changing the variable by the substitution

$$y = 2s.v \qquad (32)$$

and assuming v ≠ 0 and C ≠ 0 gives a form of Kummer's Equation [Abramowits & Stegun, 1972]:

$$y.d^2P/dy^2 - (k+y).dP/dy + \{(2k-F/v)/4\}.P = 0 \qquad (33)$$

which has as a solution the Confluent Hypergeometric Function:

$$P = H(\ (F/v-2k)/4,\ k,\ y\ ) \qquad (34)$$

This reduces to a finite length Laguerre polynomial if (F/v-2k)/4 is a negative integer, so let

$$(F/v-2k)/4 = -j, \quad j = 0,1,2,...,\ k = 1,2,..., \qquad (35)$$

or

$$F/v = 2k - 4j \qquad (36)$$

or

$$(\ AC + B\ )/v = 2n, \quad n = k-2j = ...,-2,-1,0,1,2,... \qquad (37)$$

A unique Ground State exists when n = 0 (i.e. when k = 2j ):

$$AC + B = 0 \qquad (38)$$

or

$$(\ G.M/c^2\ )(\ 2m.E/h^2\ ) + 2GMm^2/h^2 = 0 \qquad (39)$$

or (aha) :

$$E = -mc^2 \qquad (40)$$

The general states are found by solving:

$$Av^2 + 2nv - B = 0 \qquad (41)$$

so

$$v = [\ -n \pm (n^2+AB)^{1/2}\ ]/A = -(n/A).[\ 1 \pm (1 + AB/n^2)^{1/2}\ ] \qquad (42)$$

If the quantity $AB/n^2$ is much smaller than 1, the square root may be expanded as a series, and the higher powers of this quantity truncated, giving

$$v \approx -(n/A)\{\ 1 \pm [1 + AB/(2.n^2)\ ]\ \} \qquad (43)$$

so that

$$v_+ \approx -(2n/A) \qquad (44)$$

and

$$v_- \approx B/(2n) \qquad (45)$$

From equation (25) the corresponding energies of these states are given by

$$E = -(h.v)^2/(2m) \qquad (46)$$

giving

$$E_+ \approx -[\ 2/m\ ].(h.n/A)^2 \qquad (47)$$

and

$$E_- \approx -[\ 1/(8m)\ ].(h.B/n)^2 \qquad (48)$$

Expanding the quantities A and B from equations (15) and (16) gives

$$E_{+1} \approx -(2/m).[\ (h.c^2)/(GM)\ ]^2 \qquad (49)$$

and

$$E_{-1} \approx -(m/2).[\ GMm/h\ ]^2 \qquad (50)$$

**ORBITS**

To gauge the size of the orbit, we may find an approximation to the radius of the peak value ($r_p$) of the radial function $\psi$ : the Modal Orbital Radius. We shall use:

$s = r - A$ (from equation (20))

and

$A = GM/c^2$ (from equation (15))

and

$R(s) = Q(s).e^{-s.v}$ (from equation (24))

and

$Q(s) = s^k.P(s)$ (from equation (31))

and

$P(y) = H(\ (F/v-2k)/4,\ k,\ y)$ (from equation (34))

and

$y = 2s.v$ (from equation (32))

and

$v^2 = -C = -2mE/h^2$ (from equation (25))

and

$(F/v-2k)/4 = -j,\ j = 0,1,2,...$ (from equation (35))

and

$n = k-2j$ (from equation (37))

So: for example, taking the Ground State: $E = -mc^2$ :

$$n = 0,\ j = 1,\ k = 2, \qquad (51)$$

and

$$(F/v-2k)/4 = -j = -1, \qquad (52)$$

then P(y) is the Laguerre polynomial:

$$P(y) = L_1 = 1 - y \qquad (53)$$

so

$$Q(s) = s^2.(1-2v.s) \qquad (54)$$

so

$$R(s) = s^2.(1-2v.s).e^{-v.s} \quad (55)$$

and

$$v = (-C)^{1/2} = (-2mE)^{1/2}/h \quad (56)$$

The radial function R(s) of ψ has two zeroes, at s = 0, and s = 1/(2v), after which it has an oscillation decaying exponentially to zero at s = ∞ . So the peak value of R will be somewhere around halfway between the two zeroes, at approximately

$$s_p \approx 1/(4v) = (h/4)/(-2mE)^{1/2} = h/(mc.(32)^{1/2}) \quad (57)$$

so that

$$r_p \approx h/(5.7 \text{ x } mc) + GM/c^2 \quad (58)$$

or

$$r_p \approx (0.64 \text{ x } 10^{-43})/m + (0.74 \text{ x } 10^{-27}).M \quad (59)$$

**MASS DEPENDENCES**

The dependence of the orbital radius and energy levels on the masses m and M of the objects offers some interesting limitations on Gravitational Atoms.

For the Ground State (n=0) the energy dependence on the masses can be found using equation (40) :

$$E_0 = -mc^2 = -10^{+17}m \text{ Joules} \quad (71)$$

For two objects in this Ground State to be separate entities, their orbital radius needs to lie outside the sum of their respective Schwartzchild Radii, so using equation (58), this requires that :

$$r_p \approx h/(4mc) > r_{sM} + r_{sm} \quad (72)$$

so

$$h/(4mc) > 2G(m+M)/c^2 \quad (73)$$

or if M ≈ m,

$$(M.m)^{1/2} < [hc/(16.G)]^{1/2} \approx 5 \text{ x } 10^{-9} \text{ Kgm} \quad (74)$$

If the masses are greater than this limit, their Schwartzchild Radii would overlap in the Ground State, and it may be more appropriate to treat them as a single object.

The energies of the next higher levels (n = ± 1) found using equation (49) and (50) as long as A.B << 1, where

$$A.B = [\ 2G^2/(h.c)^2\ ].(M.m)^2 \quad (75)$$

so the condition that A.B << 1 requires

$$(M.m)^{1/2} << (h.c/G)^{1/2} \approx 10^{-8} \text{ Kgm} \quad (76)$$

which is guaranteed if condition (74) is satisfied.

The energy dependences on the masses below this limit are then :

$$E_{+1} \approx -(2/m).[ (h.c^2)/(GM) ]^2 \approx -2.[ (h.c^2)/G ]^2 / (m.M^2) \approx -10^{-14} / (m.M^2) \quad (77)$$

and

$$E_{-1} \approx -[ h^2/(8m) ].[ (2GMm^2/h^2 ]^2 \approx -(1/2).[ G^2/h^2 ].m^3.M^2 \approx -10^{+47}.m^3.M^2 \quad (78)$$

Taking a pair of masses near the upper limit of condition (74), say

$$M = 10^{-8} \text{ and } m = 10^{-10} \text{ Kgm} \quad (79)$$

then

$$E_{+1} \approx -10^{+12} \text{ Joules} \quad (80)$$

and

$$E_{-1} \approx -10 \text{ Joules} \quad (81)$$

**SIMILAR OBJECTS**

Some general limits can be placed on the level $E_{\pm1}$ states in the gravitational interaction between pairs of objects with similar masses.

The Ground State of systems composed of two similar objects heavier than the ~ $10^{-9}$ Kgm limit would have orbits that are substantially inside their respective Schwartzchild Radii, Continuing to treat these as a double objects seems hardly appropriate for this situation.

Consideration of level $E_{\pm1}$ states also discloses some limits.

The assumption that the total mass energy of the universe is ~ $10^{+69}$ Joules [Behr, 2007] sets some limits on Gravitational Atoms. Systems composed of two similar objects lighter than ~ $10^{-21}$ Kgm, will have an $E_{+1}$ value greater in magnitude than the mass of the observable universe. Similarly, systems composed of two similar objects heavier than ~ $10^{6}$ Kgm, will have an $E_{-1}$ value greater in magnitude than the mass of the observable universe.

Systems composed of two similar objects lighter than ~ $10^{-8}$, will have a negative $E_{-1}$ value greater than the mass of composing objects, and so that state may appear to have a negative mass. Similarly, systems composed of two similar objects heavier than ~ $10^{-13}$ Kgm, will have a negative $E_{+1}$ value greater than the mass of composing objects, and so that state may appear to have a negative mass. Thus systems composed of similar objects in the range $10^{-13}$ to $10^{-8}$ Kgms will have both $E_1$ energies greater than the mass of the composing objects, and so may exhibit negative masses in both states.

**DISCUSSION**

The solutions obtained above for the Ground State (n=0) of our Gravitational Atom, where the value of $E = -mc^2$, are of the form where the rest mass of the lighter object is totally cancelled by the negative potential energy of the state it is in. In this state, to an outside observer: the mass of the orbiting object will seem to have disappeared, and the system will exhibit only a mass equal to that of the heavier object. This is similar to the result that was obtained in the Introduction when considering Zero-Energy Radius of an object. One might have thought the kinetic energy of the orbiting second object in the Gravitational Atom would contribute positively to the energy of this state, but curiously it does not seem to.

One may expect the system still to have the combined charge of both participating objects, and also have an orbital angular momentum k, an integer, but this momentum would be about an origin at a distance from the centre of the larger object equal to half its Schwatzchild Radius. The relation of this off-centre spin to the spin of the participating objects is unclear.

This study is simplistic in its treatment of relativity and gravitation, neglecting as it does any internal structure to the objects, and the other three forces of nature, spin, and charge, but some of the qualitative results might carry through to more sophisticated and realistic models.

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**REFERENCES**

Abramowitz, M., and Stegun, I.A. (eds.), 1972, "Handbook of Mathematical Functions", Dover, New York, 9th Printing, p. 504.

Allen, C.W., 1964, "Astrophysical Quantities", Athlone Press, London, 2nd Edition.

Behr, B.B., 2007. "Universe", in Volume 19, "Encyclopedia of Science and Technology", McGraw-Hill, New York, 10th Edition, pp. 80-89.

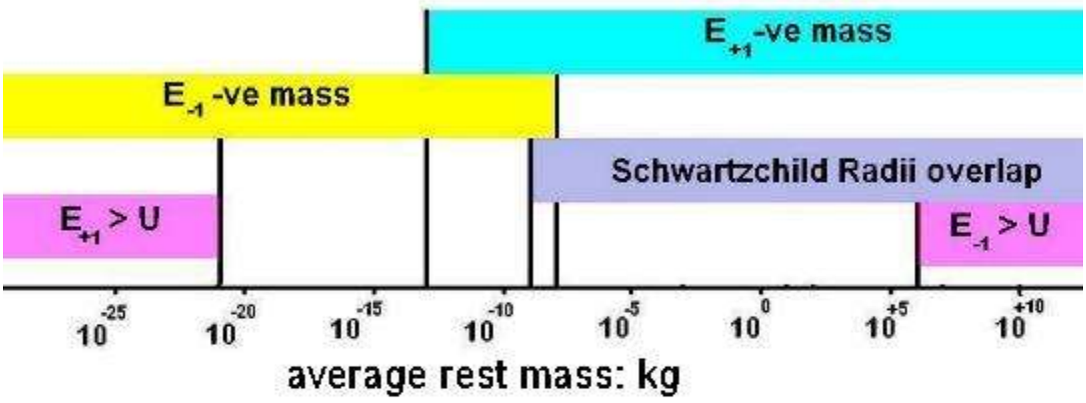
De Volp, A., 2007. "Hydrogen Bomb", in Volume 8, "Encyclopedia of Science and Technology", McGraw-Hill, New York, 10th Edition, pp. 712-713.

Fry, E.S., 2001. "Neutron", in Volume 14, "World Book", World Book, Chicago, 2001 Edition, p. 155.

Houston, W.V, 1959, "Principles of Quantum Mechanics", Dover, New York, pp. 58-82.

Schwartz, H.M., 2007. "Relativistic Mechanics", in Volume 15, "Encyclopedia of Science and Technology", McGraw-Hill, New York, 10th Edition, pp. 330-332.

**APPENDIX 1**




Graphical representation of the constraints on the two first level excited states

## APPENDIX 2

| Primary Variables & Constants<br>MKS units [Allen, 1964] | Equations defining<br>Intermediate Variables |
|---|---|
| c = velocity of light = $3.0 \times 10^{8}$, m/s | A equation (15) |
| E = system energy, J | B equation (16) |
| G = universal constant of gravitation = $6.7 \times 10^{-11}$, $m^3kg^{-1}s^{-2}$ | C equation (17) |
| h = reduced Planck's constant = $1.1 \times 10^{-34}$, J.s | F equation (19) |
| m = rest mass of the light object, kg | H equation (34) |
| M = rest mass of the heavy object, kg | j equation (35) |
| r = separation of the two objects, m | k equation (23) |
| u = travelling mass of the light object, kg | L equation (53) |
| U = mass energy of the visible universe, J | n equation (37) |
| V = potential energy, J | P equation (34) |
| T = kinetic energy, J | R equation (22) |
| | Q equation (27) |
| | s equation (20) |
| | v equation (25) |
| | W equation (22) |
| | y equation (32) |
| | Δ equation (11) |
| | ψ equation (22) |

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