# A Bit of Secrecy for Gaussian Source Compression

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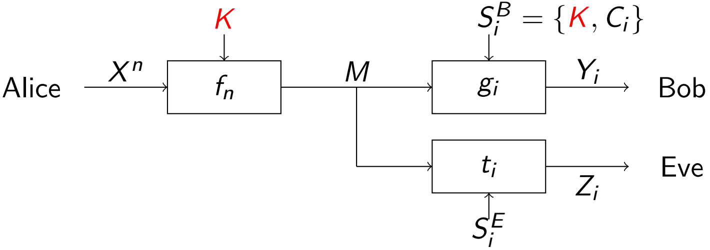
# Problem Setup

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- $K \sim Unif[1 : 2^{nR_s}]$

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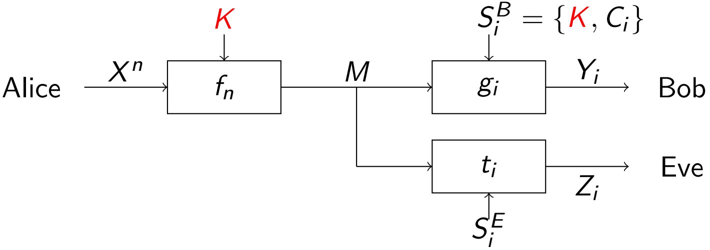


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- $K \sim Unif[1 : 2^{nR_s}]$
- Encoder $f_n : \mathcal{X}^n \times \mathcal{K} \mapsto \mathcal{M}$, $\mathcal{M} = [1 : 2^{nR}]$
- Bob's decoder $\{g_i : \mathcal{M} \times \mathcal{S}_i^B \mapsto \mathcal{Y}\}_{i=1}^n$, $C_i = \{X^{i-1}, Y^{i-1}, Z^{i-1}\}$
- Eve's decoder $\{t_i : \mathcal{M} \times \mathcal{S}_i^E \mapsto \mathcal{Z}\}_{i=1}^n$

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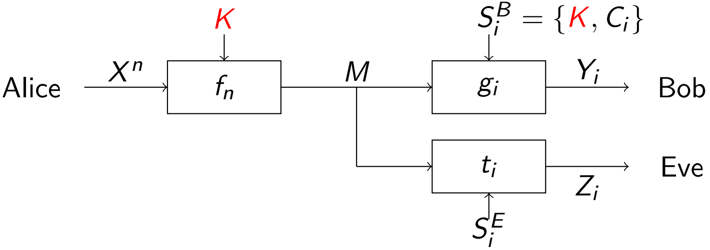


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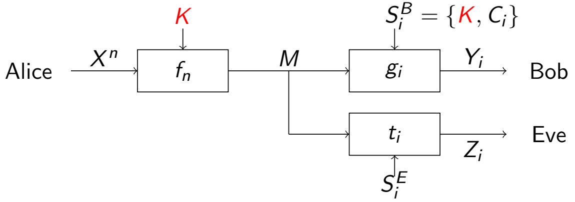
- Eve's decoder $\{t_i : \mathcal{M} \times \mathcal{S}_i^E \mapsto \mathcal{Z}\}_{i=1}^n$
- Payoff $\pi(x, y, z) \triangleq \frac{1}{\sigma_0^2}[(z - x)^2 - (y - x)^2]$
- $\pi(x^n, y^n, z^n) \triangleq \frac{1}{n}\sum_{i=1}^n \pi(x_i, y_i, z_i)$

# Three Cases



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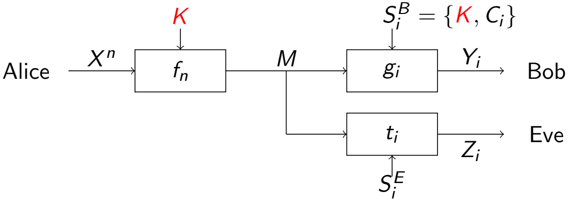


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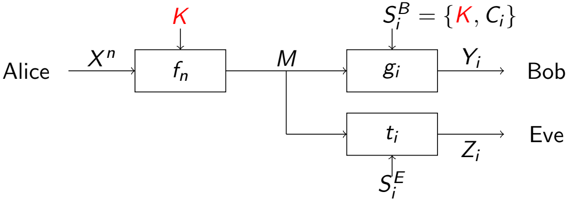


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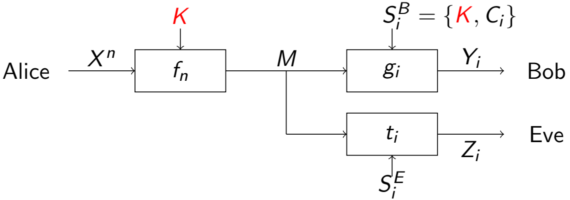


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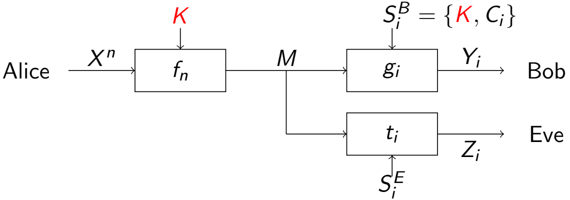


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## Definition

*A secrecy rate-payoff triple* $(R, R_s, \Pi)$ *is achievable if* $K \in [1 : 2^{nR_s}]$, $M \in [1 : 2^{nR}]$, *and*

$$\lim_{n\to\infty} \sup_{\{f_n, \{g_i\}_{i=1}^n\}} \inf_{\{t_i\}_{i=1}^n} \mathbb{E}\pi(X^n, Y^n, Z^n) \geq \Pi.$$

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- Weak eavesdropper
  - General source
  - General distortion measure
  - Lossless compression
  - ✓ $R_s > 0$, $R > H(X)$, $D_{\max} \triangleq \min_{\hat{x}} \mathbb{E} d(X, \hat{x})$ [Schieler & Cuff '12]

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- Goal: optimize for Gaussian source

# Why do we consider this model?

- Gaussian source
- Tradeoff between lossy compression rate and secrecy
- Causal disclosure [Schieler & Cuff '13]

# Weak Eavesdropper

**Theorem**

*The secrecy rate-payoff triple $(R, R_s, \Pi)$ for a* **weak eavesdropper** *is achievable for an i.i.d. Gaussian source if and only if*

$$R_s > 0, \ \textit{and}$$

$$\Pi \leq 1 - \exp(-2R).$$

- Maximum distortion between Alice and Eve
- Distortion-rate function for Gaussian source $R(D) = \frac{1}{2} \log \frac{\sigma_0^2}{D}$

# Causal Source Awareness

- General solution to any i.i.d. source sequence and payoff function:

$$
\begin{aligned}
\Pi_{p_0}(R, R_s) &= \max_{p(y,u|x)\in\mathcal{P}} \min_{z(u)} \mathbb{E}\pi(X, Y, z(U)) \\
\mathcal{P} &= \left\{ \begin{array}{l} p(y,u|x): \\ R_s \geq I(X; Y|U) \\ R \geq I(X; U, Y) \end{array} \right\}.
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&= \frac{1}{\sigma_0^2} \max_{p(y,u|x)\in\mathcal{P}} \left[ \sum_{x,u} p(u|x)p_0(x)(x - \mathbb{E}[X|U = u])^2 \right. \\
&\quad \left. - \sum_{x,y} p(y|x)p_0(x)(y - x)^2 \right]
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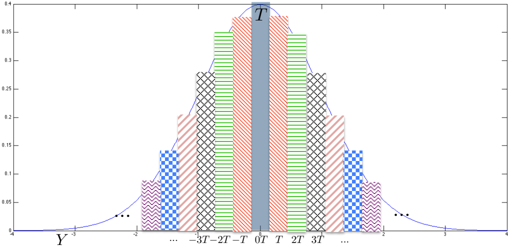


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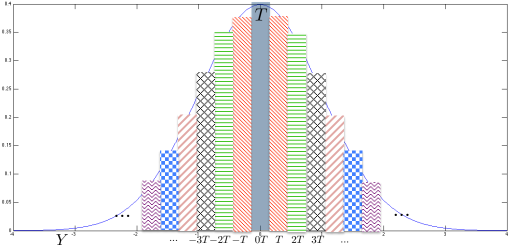


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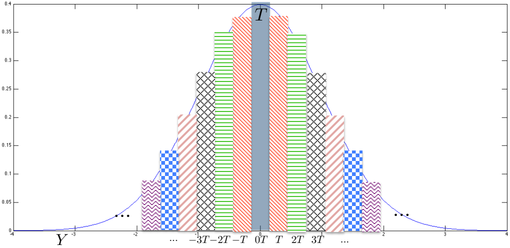


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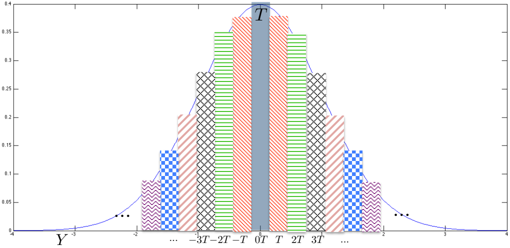


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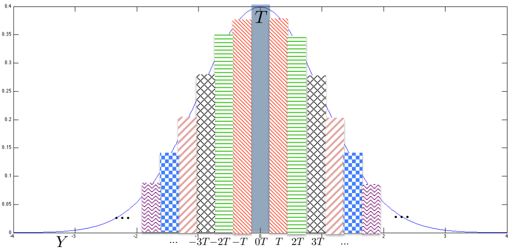


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- Summarizing:
  $\Pi_{p_0}(R, R_s) \geq 1 - \frac{\pi e}{2} 2^{-2R}$
  for $R_s \geq 1$ bit and $R \to \infty$.

# Optimal Payoff for $R_s \geq 1$ bit

**Theorem**

*If the key rate $R_s \geq 1$ bit, the optimal secrecy rate-payoff function for an i.i.d. Gaussian source and* **causal source awareness** *is given by*

$$\Pi_{p_0}(R, R_s) = 1 - 2^{-2R}.$$

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- Recall:
  - $\frac{1}{\sigma_0^2} \max_{p(y,u|x)\in\mathcal{P}} \Big[ \sum_{x,u} p(u|x)p_0(x)(x - \mathbb{E}[X|U=u])^2$
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  - $R_s \geq I(X;Y|U)$
  - $R \geq I(X;U,Y)$
- Relax constraints on rates $R$ and $R_s$
- $\max_{p(y,u|x)} \sum_{u,x} p(u|x)p_0(x)(x - \mathbb{E}[X|U=u])^2 = \sigma_0^2$
- $\min_{p(y,u|x):I(X;Y)\leq R} \sum_{x,y} p(y|x)p_0(x)(y-x)^2 = \sigma_0^2 2^{-2R}$

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- $U \triangleq |Y|$
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- Constraint on $R$: $I(X; Y, U) = I(X; Y)$
- Constraint on $R_s$: $I(X; Y|U) = I(X; V|U) < 1$ bit
- $\mathbb{E}[X|U = u] = \frac{1}{2}\mathbb{E}[X|Y = u] + \frac{1}{2}\mathbb{E}[X|Y = -u] = 0$, for all $u$

$\Rightarrow \Pi_{p_0}(R, R_s) = 1 - 2^{-2R}$ for $R_s \geq 1$ bit

# Causal General Awareness

- General solution to any i.i.d. source sequence and payoff function:

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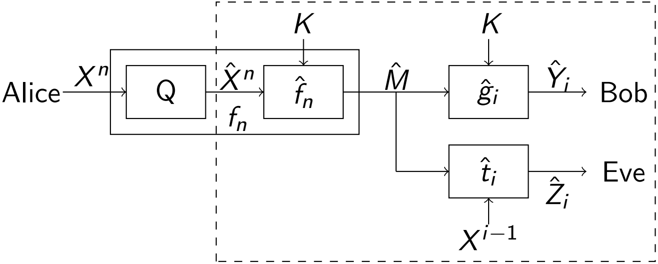
# What do we do for $R_s < 1$ bit?

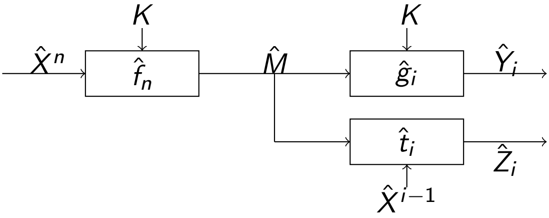
- Optimization too hard
- What if we quantize the source sequence symbol-by-symbol?

# Quantization Special Case for Causal Source Awareness




- Alice quantizes $X^n$ symbol-by-symbol $\hat{X}^n$ before transmission
- Bob reproduces the scalar quantization version $\hat{X}^n$
- $\hat{X}_i = \mathbb{E}[X_i|\text{Quantization bin of } X_i]$
- What is the optimal payoff function $\Pi^{\Delta}_{p_0}(R, R_s)$?

# Lossless Compression




## Definition

*The rate-distortion triple* $(R, R_s, D)$ *is achievable if*

$$\mathbb{P}[\hat{Y}^n \neq \hat{X}^n] \to 0 \text{ as } n \to \infty, \text{ and}$$

$$\lim_{n\to\infty} \sup_{\{\hat{f}_n, \{\hat{g}_i\}_{i=1}^n\}} \inf_{\{\hat{t}_i(\hat{m}, \hat{s}_i^E)\}_{i=1}^n} \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n (\hat{Z}_i - \hat{X}_i)^2\right] \geq D.$$

# Lossless Compression Rate Distortion

- $\hat{X} \sim \hat{p}_0$
- $\mathcal{Q} = \{p(\hat{u}|\hat{x}) : R \geq H(\hat{X}), R_s \geq H(\hat{X}|\hat{U})\}$
- $D_{\hat{p}_0}(R, R_s) \triangleq \max_{p(\hat{u}|\hat{x}) \in \mathcal{Q}} \min_{\hat{z}(\hat{u})} \mathbb{E}[(\hat{z}(\hat{U}) - \hat{X})^2]$

**Theorem 4.1 [Cuff '10]**

*$(R, R_s, D)$ is achievable iff*

$$D \leq D_{\hat{p}_0}(R, R_s)$$

# Applying Result from Lossless Compression

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**Lemma**

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**Theorem**

$\Pi_{p_0}^{\Delta}(R, R_s) = \frac{1}{\sigma_0^2} D_{\hat{p}_0}(R, R_s)$.

- $D_{\hat{p}_0}(R, R_s)$ can be calculated as a linear program.

# Numerical Result for Causal Source Awareness

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A simple quantization scheme

- Given $T$ and $N$
- $Y \triangleq nT$
- $U \triangleq n \bmod N$
- Greedily solve for optimal $T$ s.t. $R \geq I(X; U, Y)$
- Solve for optimal $N$ s.t. $R_s \geq I(X; Y|U)$

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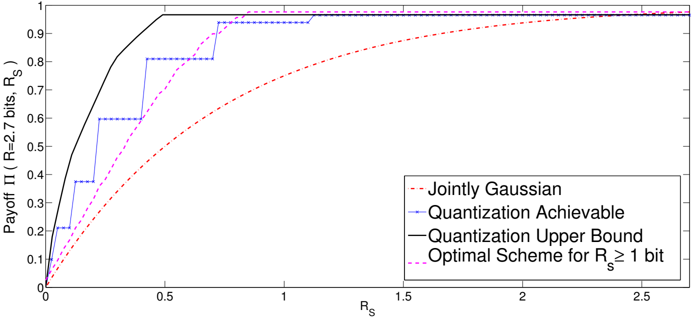

# Conclusion

- Weak eavesdropper ✓
  - $R_s > 0$
  - Bob: Rate-distortion
  - Eve: Maximum distortion
- Causal source awareness
  - Jointly Gaussian sub-optimal
  - $R_s \geq 1$ bit ✓
  - $R_s < 1$ bit symbol-by-symbol
- Causal general awareness
  - Jointly Gaussian sub-optimal
  - $R_s \geq 1$ bit ✓