أولمبياد الرياضيات الخليجي
في الدول الأعضاء بمكتب التربية العربي لدول الخليج
GMO

7^{th} Gulf Mathematical Olympiad

Muscat, Sultanate of Oman, October 6-10, 2019

The paper will last 4 hours 30 minutes

Each problem is worth 10 points

Problem 1

Let $ABCD$ be a trapezium (trapezoid) with AD parallel to BC, and J be the intersection of the diagonals AC and BD. Point P is chosen on the side BC such that the distance from C to the line AP is equal to the distance from B to the line DP. *The following three questions 1, 2 and 3 are independent, so that a condition in one question does not apply in another question.*

1. Suppose that $Area(\triangle AJB) = 6$ and that $Area(\triangle BJC) = 9$. Determine $Area(\triangle APD)$.
2. Find all points Q on the plane of the trapezium such that
$$Area(\triangle AQB) = Area(\triangle DQC).$$
3. Prove that PJ is the angle bisector of $\angle APD$.

Problem 2

1. Find N, the smallest positive multiple of 45 such that all of its digits are either 7 or 0.
2. Find M, the smallest positive multiple of 32 such that all of its digits are either 6 or 1.
3. How many elements of the set $\{1, 2, 3, \cdots, 1441\}$ have a positive multiple such that all of its digits are either 5 or 2?

Problem 3

Consider the set $S = \{1, 2, 3, \cdots, 1441\}$.

1. Nora counts those subsets of S having exactly two elements, the sum of which is even. Rania counts those subsets of S having exactly two elements, the sum of which is odd. Determine the numbers counted by Nora and Rania.

2. Let t be the number of subsets of S which have at least two elements and the product of the elements is even. Determine the greatest power of 2 which divides t.

3. Ahmad counts the subsets of S having 77 elements such that in each subset the sum of the elements is even. Bushra counts the subsets of S having 77 elements such that in each subset the sum of the elements is odd. Whose number is bigger? Determine the difference between the numbers found by Ahmad and Bushra.

Problem 4

Consider the sequence $(a_n)_{n\geq 1}$ defined by $a_n = n$ for $n \in \{1, 2, 3, 4, 5, 6\}$, and for $n \geq 7$:

$$a_n = \left\lfloor \frac{a_1 + a_2 + \cdots + a_{n-1}}{2} \right\rfloor$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. *For example :* $\lfloor 2.4 \rfloor = 2$; $\lfloor 3 \rfloor = 3$ *and* $\lfloor \pi \rfloor = 3$.
For all integers $n \geq 2$, let $S_n = \{a_1, a_2, \cdots, a_n\} \setminus \{r_n\}$ where r_n is the remainder when $a_1 + a_2 + \cdots + a_n$ is divided by 3. *The backslash* $\setminus$ *denotes the "remove it if it is there" notation. For example :* $S_4 = \{2, 3, 4\}$ *because* $r_4 = 1$ *so* 1 *is removed from* $\{1, 2, 3, 4\}$. *However* $S_5 = \{1, 2, 3, 4, 5\}$ *because* $r_5 = 0$ *and* 0 *is not in the set* $\{1, 2, 3, 4, 5\}$.

1. Determine S_7, S_8, S_9 and S_{10}.

2. We say that a set S_n, for $n \geq 6$, is *well-balanced* if it can be partitioned into three pairwise disjoint subsets with equal sum. *For example :* $S_6 = \{1, 2, 3, 4, 5, 6\} = \{1, 6\} \cup \{2, 5\} \cup \{3, 4\}$ *and* $1 + 6 = 2 + 5 = 3 + 4$. Prove that S_7, S_8, S_9 and S_{10} are well-balanced.

3. Is the set S_{2019} well-balanced? Justify your answer.