Assignment 2

Due: 7pm on October 27, 2025

This is a *mandatory* assignment for everyone with a *mandatory* submission requirement. Please submit the physical copy of your work and indicate your full name on the file. Write all your statements and deriviations as clearly as you can.

Provided all problems have been attempted properly (with a solution or a description of what you have tried), these problems weight equally in your grading.

1. **(Commodity Type)** A utility function is *homothetic* if

$$u(ax) = au(x) \text{ for all } a > 0$$

a Prove, when the utility function is homothetic and the Walrasian demand is single-valued, the demand is in the form $\xi(p, w) = g(p)w$ for some function g. [1]

b Prove that if the utility function is homothetic, then there is no Giffen good.

c Explain briefly why you would or would not expect utility functions to be homothetic.

Proof. For a, by definition, $u(\xi(p, w)) \geq u(x)$ for any x such that $p \cdot x \leq w$. By u is homothetic, we have $u(a\xi(p, w)) \geq u(ax)$ for any $a > 0$. Equivalently, we have $u(a\xi(p, w)) \geq u(y)$ for any y such that $p \cdot x \leq y$. That is, $a\xi(p, w) = \xi(p, aw)$, for all $a > 0$. Hence,

$$\xi(p, w) = w\xi(p, 1) = g(p)w,$$

where $g(p)$ is defined to be $\xi(p, 1)$.

For b, by (a), we note $\frac{d\xi_i}{dw}(p, w) = g_i(p) \geq 0$ for any i, as $\xi(p, 1) \geq 0$. Thus, by the Slutsky decomposition, we have

$$\frac{d\xi_i}{dp_i} = \frac{dh_i}{dp_i} - \xi_i \frac{d\xi_i}{dw} \leq 0,$$

as h_i is downward sloping by the law of demand.

For c, we note in the proof of b, we used the observation that all goods are normal under homothetic utility functions. But I am convinced the existence of inferior good such as low quality cherries. ∎

2. **(Consumer Welfare)** Consider a price change from the initial price p to a new price p' in which only the price of commodity i decreases. Show if commodity i is inferior, compare

[1] Due to this property, homothetic utility functions are very convenient in some applications.

the compensating variation (CV) and the equivalence variation (EV).[2]

Proof. We show the CV is larger than EV.

First, we show $e(p, v)$ nondecreasing in p: Recall that $e(p, v) = p \cdot h(p, v) \leq p \cdot x$, for any x such that $u(x) \geq v$. Let $p' > p$ in $\mathbb{R}^n_+$, then, we have

$$e(p', v) = p' \cdot h(p', v) \geq p \cdot h(p', v) \geq e(p, v),$$

where the second last inequality is by $p' > p$ and the last inequality is by taking $x = h(p', v)$ in the definition.

Now, we see $e(p, v)$ is strictly increasing in v. We prove by contradiction. Suppose $e(p, v') \leq e(p, v)$ for some $v' > v$, we have $u(h(p, v')) = v' > v$. Thus, shrinking a bit of consumption will reduce the expenditure: for $\lambda < 1$ but sufficiently close to 1, we have $u(\lambda h(p, v')) > v$. Thus, $e(p, v) \leq p \cdot \lambda h(p, v') < e(p, v')$.

To show the claim, we recall that

$$CV = \int_{p_i'}^{p_i} h_i(p, v) dp_i,$$

$$EV = \int_{p_i'}^{p_i} h_i(p, v') dp_i,$$

To show $CV > EV$, we just need to show $h_i(p, v') < h_i(p, v)$ for all p. Equivalently, we show

$$\xi_i(p, e(p, v')) < \xi_i(p, e(p, v)).$$

Since the new price p' is less than the old price p, we know that $v < v'$. Thus, $e(p, v') > e(p, v)$. Since i is inferior, we have $\xi_i(p, e(p, v')) < \xi_i(p, e(p, v))$, and finishes the proof. ■

3. **(Preference)**

   a. Prove if a relation is complete, it must be reflexive.

   b. Provide an example of preference that is reflexive but incomplete.

   c. Is the Lexicographic preference on $\mathbb{R}^2_+$ convex?

Proof. For a, take any $x = y$ in X. By completeness of R, either xRy or yRx. Therefore, xRx holds for any $x \in X$.

[2] You would need to use and prove that $e(p, v)$ is strictly increasing in v and the equivalence between two types of demand here.

For b, the binary relation $=$ on any non-singleton X is reflexive but incomplete.

For c, yes, it is convex. To see this, given any $(x_1, x_2) \in \mathbb{R}^2$, the better than set is:

$$B = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 > x_1 \text{ or } (y_1 = x_1 \text{ and } y_2 \geq x_2)\}.$$

Take any $(y_1, y_2), (y_1', y_2') \in B$, we check for

$$(z_1, z_2) = \lambda(y_1, y_2) + (1 - \lambda)(y_1', y_2'), \lambda \in [0, 1].$$

We discuss by cases:

Case 1 $y_1 > x_1, y_1' > x_1$ then, $z_1 > x_1$, so $(z_1, z_2) \in B$.

Case 2 $y_1 > x_1, y_1' = x_1, y_2' \geq x_2'$ then $z_1 > x_1$, so $(z_1, z_2) \in B$.

Case 3 $y_1 = x_1, y_2 \geq x_2, y_1' > x_1$, then $z_1 > x_1$, so $(z_1, z_2) \in B$.

Case 4 $y_1 = x_1, y_2 \geq x_2, y_1' = x_1, y_2' \geq x_2$. Then, $z_1 = x_1, z_2 \geq x_2$. So $(z_1, z_2) \in B$.

Thus, B is convex. ■

4. **(Utility Representation)** Write whether the following statements are true or false, and justify your answers:

a. All continuous preferences do not have discontinuous utility representations.

b. Some continuous preference does not have a discontinuous utility representation.

c. All continuous preferences can be represented by a utility function, which has a range given by $[a^*, b^*]$ for $a^* < b^*$.

Solution. For a, it is false. Consider a preference $\succeq$ on $\mathbb{R}$ with $x \succeq y$ whenever $x \geq y$. It is continuous, but admits a discontinuous utility representation $u : \mathbb{R} \to \mathbb{R}$ with $u(x) = x$ for $x \geq 0$ and $u(x) = x - 1$ for $x < 0$.

For b, it is true. Suppose the preference is a constant preference: for all $x, y \in X$, $x \sim y$. Then, the only utility representation must be a constant utility representation. It admits no discontinuous utility (which necessarily implies some alternative is better than some other).

For c, by Debreu's theorem, there exists a continuous utility representation $u : X \to \mathbb{R}$. The range of u must be an nontrivial interval $[a, b]$ given the preference is nontrivial. In particular, we allow a, b to be $-\infty$ or ∞. Next, we can simply argue that there exists an increasing transformation that transform this interval $[a, b]$ to the target $[a^*, b^*]$. There are four cases.

Case 1 When all a, b, a^*, b^* are finite, this can be done by a positive affine transformation whenever $a \neq b$: take $\Phi : \mathbb{R} \to \mathbb{R}$ be $\Phi(x) = a^* - \frac{a(b^*-a^*)}{b-a} + \frac{b^*-a^*}{b-a}x$, we have $\Phi([a, b]) = [a^*, b^*]$. When $a = b$, the preference is trivial, simply represents it to any number in $[a^*, b^*]$ will do the work.

Case 2 When $a = -\infty$ and $b < \infty$, take $\Phi : \mathbb{R} \to \mathbb{R}$ be $\Phi(x) = a^* + \frac{b^*-a^*}{e^b}e^x$, we have $\Phi([a, b]) = [a^*, b^*]$. So $\Phi(u)$ is a utility representation satisfies the requirement.

Case 3 When $a > -\infty$ and $b = \infty$, take $\Phi : \mathbb{R} \to \mathbb{R}$ be $\Phi(x) = b^* + \frac{a^*-b^*}{e^{-a}}e^{-x}$, we have $\Phi([a, b]) = [a^*, b^*]$, and Φ is increasing. So $\Phi(u)$ is a utility representation satisfies the requirement.

Case 4 When $a = -\infty$ and $b = \infty$, take $\Phi : \mathbb{R} \to \mathbb{R}$ be $\Phi(x) = \frac{a^*+b^*}{2} + \frac{b^*-a^*}{\pi} \arctan(x)$, we have $\Phi([a, b]) = [a^*, b^*]$, and Φ is increasing. So $\Phi(u)$ is a utility representation satisfies the requirement.

Note, here I showed only the range of the utility presentation can be designed. One can show that the image of utility presentation can be designed in the same manner. ■