Assignment 3

Due: 7pm on November 10, 2025

This is a *mandatory* assignment for everyone with a *mandatory* submission requirement. Please submit the physical copy of your work and indicate your full name on the file. Write all your statements and deriviations as clearly as you can.

Provided all problems have been attempted properly (with a solution or a description of what you have tried), these problems weight equally in your grading.

1. **(Debreu's Theorem)** A binary relation $\succeq$ on $\mathbb{R}^2_+$ defines a preference as follows: for any pairs of vectors $(x, y), (x', y') \in \mathbb{R}^2_+$,

- if $x \neq y$ and $x' \neq y'$, $(x, y) \succeq (x', y')$ if and only if $xy \geq x'y'$;
- if $x = y$ but $x' \neq y'$, $(x, y) \succeq (x', y')$ if and only if $xy \geq x'y' - 1$;
- if $x \neq y$ but $x' = y'$, $(x, y) \succeq (x', y')$ if and only if $xy \geq x'y' + 1$;
- if $x = y$ and $x' = y'$, $(x, y) \succeq (x', y')$ if and only if $x \geq x'$.

Answer the following questions:

a. Is $\succeq$ defined above rational?

b. Check if $\succeq$ is continuous.

c. Find a utility representation of $\succeq$.[1]

d. Is $\succeq$ convex?

Proof. For (a), for every $(x, y), (x', y') \in \mathbb{R}^2_+$, if $x \neq y$ and $x' \neq y'$, then either $xy \geq x'y'$ or $x'y' \geq xy$. Therefore, the two vectors are comparable under $\succeq$. When $x = y$ but $x' \neq y'$, then either $xy \geq x'y' - 1$ or $x'y' > xy + 1 \geq xy - 1$. Therefore, two vectors are comparable. When $x = y$ and $x' = y'$, then either $x \geq x'$ or $x' \geq x$. The two vectors are comparable. Therefore, $\succeq$ is complete.

For transitivity, suppose $(x, y) \succeq (x', y')$ and $(x', y') \succeq (x'', y'')$. We discuss by cases:

- If $x = y$,

[1]If your answer in either b or c is $\succeq$ is discontinuous, it means a discontinuous preference could have a utility representation (though maybe discontinuous).

  - if $x' = y'$, then $x \geq x'$. If $x'' = y''$, then $x' \geq x''$, which implies $x \geq x'$, which implies $(x, y) \succeq (x'', y'')$. If $x'' \neq y''$, then $x'y' \geq x''y'' - 1$. But $x \geq x'$ so $xy = x^2 \geq (x')^2 = x'y'$. Therefore, $xy \geq x''y'' - 1$. Therefore, $(x, y) \succeq (x'', y'')$.
  - If $x' \neq y'$. Then, $xy \geq x'y' - 1$. If $x'' = y''$, then $x'y' \geq x''y'' + 1$, which implies $xy \geq x''y''$, and $(x, y) \succeq (x'', y'')$. If $x'' \neq y''$, we have $x'y' \geq x''y''$. Therefore, $xy \geq x''y'' - 1$, which implies $(x, y) \geq (x'', y'')$.

- If $x \neq y$,
  - if $x' = y'$, then $x \geq x'$. If $x'' = y''$, then $xy \geq x'y' + 1$. If $x'' \neq y''$, then $x'y' \geq x''y'' - 1$. Then, $xy \geq x''y''$, which implies $(x, y) \succeq (x'', y'')$. If $x'' = y''$, we have $x' \geq x''$. Therefore, $xy \geq x''y'' + 1$, which implies $(x, y) \succeq (x'', y'')$.
  - If $x' \neq y'$. Then, $xy \geq x'y'$. If $x'' = y''$, then $x'y' \geq x''y'' + 1$, which implies $xy \geq x''y'' + 1$, and $(x, y) \succeq (x'', y'')$. If $x'' \neq y''$, we have $x'y' \geq x''y''$. Therefore, $xy \geq x''y''$, which implies $(x, y) \geq (x'', y'')$.

Hence, $\succeq$ is transitive. Thus it is rational.

For (b), take $v^n = (1.5, 1)$ and $w^n = (1 + 1/n, 1 - 1/n)$. Each $v^n \succeq y^n$ because $1.5 \cdot 1 > 1 - 1/n^2$. However, in the limit $(1.5, 1) \not\succeq (1, 1)$. Thus, $\succeq$ is not continuous.

For (c), the utility function is given by

$$u(x, y) = \begin{cases} xy & x \neq y \\ xy + 1 & x = y \end{cases}.$$

For (d), take $v = (1.5, 1)$ and $w = (1, 1)$ with $w \succeq v$ by definition. But $\frac{v+w}{2} = (1.25, 1) \not\succeq v$. That is, $\succeq$ is not convex. ■

2. (**Expected Utility**) For a non-singleton outcome set $C \subset \mathbb{R}$, check if the following function is von Neumann-Morgenstern utility function:

$$U(p) = \mathbb{E}(p) - Var(p).$$

Here, p is a lottery on C, the expectation $\mathbb{E}(p) = \sum_{c \in C} p(c)c$ and the variance $Var(p) = \sum_{c \in C} p(c)(c - \mathbb{E}(p))^2$. If you think it is, prove it by finding the Bernoulli utility function. If you think it is not, prove your claim.

Proof. To see U is not a von Neumann-Morgenstern utility function, we prove it violates independence. Let p^1 be the lottery having a value 2 with a probability 1/2 and having a value 0 with a probability 1/2, p^2 be the deterministic lottery having a value 0 with a probability 1, p^3 be the deterministic lottery having a value 1 with a probability 1.

Thus, $U(p^1) = 1 - 1 = 0 = U(p^2)$. Therefore, $p^1 \sim p^2$.

Let $\lambda = 1/2$. We have $\lambda p^1 + (1-\lambda)p^3$ is a lottery having a value 2 with a probability 1/4, having a value 1 with a probability 1/2, having a value 0 with a probability 1/4. Therefore, $\mathbb{E}(\lambda p^1 + (1-\lambda)p^3) = 1$ and $Var(\lambda p^1 + (1-\lambda)p^3) = \frac{1}{2}$. Thus, $U(\lambda p^1 + (1-\lambda)p^3) = \frac{1}{2}$.

Meanwhile, $\lambda p^2 + (1-\lambda)p^3$ is a lottery having a value 1 with a probability 1/2, having a value 0 with a probability 1/2. Therefore, $\mathbb{E}(\lambda p^2 + (1-\lambda)p^3) = 1/2$ and $Var(\lambda p^2 + (1-\lambda)p^3) = \frac{1}{4}$. Thus, $U(\lambda p^2 + (1-\lambda)p^3) = \frac{1}{4} < U(\lambda p^1 + (1-\lambda)p^3)$.

Therefore, $\lambda p^1 + (1-\lambda)p^3 \succ \lambda p^2 + (1-\lambda)p^3$, violating the independence axiom. Thus, U is not vNM.

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3. **(Risk Attitude)** Consider two agents who use von Neumann-Morgenstern utility function to evaluate uncertainties. Their preferences are parametrized by two Bernoulli utility functions on $\mathbb{R}_+$:

$$u(w) = \log w$$

$$v(w) = (w-1)^3$$

For the uncertainty, suppose only the case that all uncertainties are generated by a coin flip. That is, there are two states - Head (H) and Tail (T), each of which happens with probability 0.5. In this situation, a random variable is denoted by two real numbers $x(H), x(T)$.

a. Derive the certainty equivalence for a random variable $x = (x(H), x(T))$ for both agents.

b. Write down the Arrow-Pratt index for both agents.

c. Could you try to compare the risk attitude of two agents?

Proof. For a, the certainty equivalence of x is defined by a deterministic random variable with outcome $c(x)$ such that $c(x) \sim x$. When the Bernoulli utility function is $u(x)$,

$$\log(c(x)) = 0.5\log(x(H)) + 0.5\log(x(T)) \iff c(x) = \sqrt{x(H)x(T)}.$$

When the Bernoulli utility function is $v(x)$,

$$(c(x)-1)^3 = 0.5(x(H)-1)^3 + 0.5(x(t)-1)^3 \iff c(x) = 1 + \sqrt[3]{0.5(x(H)-1)^3 + 0.5(x(t)-1)^3}.$$

For b, the Arrow-Pratt index is defined by $A_u(w) = -\frac{u''(w)}{u'(w)}$. Thus, for $u(w) = \log(w)$, $A_u(w) = \frac{1}{w}$. For $v(w) = (w-1)^3$, $A_v(w) = \frac{2}{1-w}$. Note $v'(1) = 0$ but $v''(1) > 0$. Therefore, the Arrow-Pratt index of v is undefined when $w = 1$. Another way to see its nonexistence the limit from two sides of 1 are different for $A_v(1)$.

For c, we compare the Arrow-Pratt indices. When $w < 1$, we have

$$A_u(w) > A_v(w) \iff \frac{1}{w} > \frac{2}{1-w} \iff w < 1/3.$$

Therefore, on $(0, \frac{1}{3})$, the log agent is more risk averse; on $(\frac{1}{3}, 1)$, the cubic agent is more risk averse. When $w > 1$, the cubic agent is risk loving with $A_v(w) > 0$. Thus the risk averse log agent is more risk averse than the cubic agent.

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4. **(Ambiguity)** In this example, we try to give an explanation of a seemingly paradoxical experiment by assuming a decision maker may have multiple beliefs.

Consider two boxes A and B. Each box contains 100 balls. The balls are either white or black. Box A contains 51 white balls and 49 black balls. The percentage of white and black balls in Box B is unknown. The participant of this experiment is asked to choose one box among these two boxes. Once a box is chosen, a ball will be uniformly drawn from this box, and the color of this ball will be announced.

This participant will be asked to choose a box in the following two cases in order.

Case 1: the award is 100 dollars when the drawn ball is white, and 0 otherwise.

Case 2: the award is 100 dollars when the drawn ball is black, and 0 otherwise.

In the experiment, most participants choose Box A in both cases. Now, we analyze this experiment formally.

A *belief* on the color of the chosen ball in box B can be represented by a number $\pi \in [0, 1]$, where π is the probability that the chosen ball is white. Also, we normalize the Bernoulli utility function u defined on $\{0, 100\}$ by $u(0)$=0,u(100) $= 1$.

Now, we assume that this decision maker holds multiple beliefs. The set of his beliefs is represented by a set $P \subset [0, 1]$. Consider the following utility function over actions A and B: For case 1, $U_1 : \{A, B\} \to \mathbb{R}$ is defined by

$$U_1(A) = 0.51, U_1(B) = \min\{\pi : \pi \in P\}$$

For case 2, $U_2 : \{A, B\} \to \mathbb{R}$ is defined by

$$U_2(A) = 0.49, U_2(B) = \min\{1 - \pi : \pi \in P\}$$

Namely, his utility from choice A is the expected utility of 100 dollars with the (objective) probability calculated from the distribution of white and black ball in box A. However, his utility from choice B is the expected utility of 100 dollars with the probability associated with the most pessimistic belief in P.

a. Prove that if P consists of only one belief, then U_1 and U_2 are derived from a von Neumann-Morgenstern utility function, and $U_1(A) > U_1(B)$ if and only if $U_2(A) < U_2(B)$.[2]

b. Find a set P for which $U_1(A) > U_1(B)$ and $U_2(A) > U_2(B)$.

Proof. For a, if $P = \{\pi\}$, we have $U_1(B) = \pi$ and $U_2(B) = 1 - \pi$. Hence, both U_1 and U_2 are derived from the expected utility

$$pu(100) + (1 - p)u(0).$$

When deriving U_1, p is the probability that the drawn ball is white, and when deriving U_2, p is the probability that the drawn ball is black.

Moreover, we have

$$U_1(A) > U_1(B) \iff 0.51 > \pi \iff 0.49 < 1 - \pi \iff U_2(A) < U_2(B).$$

That is, in this case, a decision maker will not choose A in both cases.

For (b), note that

$$U_1(A) > U_1(B) \iff 0.51 > \min\{\pi : \pi \in P\};$$

$$U_2(A) > U_2(B) \iff 0.49 > \min\{\pi : \pi \in P\} \iff 0.51 > \max\{\pi : \pi \in P\}.$$

That is, for every set $P \subset [0, 1]$ such that $\min P < 0.51 < \max P$, we have

$$U_1(A) > U_1(B) \text{ and } U_1(A) > U_2(B).$$

■

[2]i.e. when there is a single belief, a decision maker with a von Neumann-Morgenstern utility function will not choose box 1 in both cases.