Assignment A

Due: 7pm on September 10, 2025

This is a *mandatory* assignment for everyone with optional submission requirement: these exercises will be used later in the course, and the assumption is everyone has learn the material through working on this assignment.

If you choose to submit your work, please submit the physical copy of your work. Write all your statement and deriviations as clearly as you can. And mark your file with your unique character that is not your name. For this assignment, I will not grade any work with a student name on it. In addition, I will grade an assignment only if all problems have been properly attempted.

1. **(Set Operations)** For a price vector $p \in \mathbb{R}^N_{++}$ and a income level $w > 0$, define a budget set as $B(p, w) = \{x \in \mathbb{R}^N_+ : p \cdot x \leq w\}$. Prove, for any $\lambda > 0$, $B(p, w) = B(\lambda p, \lambda w)$.[1]

2. **(Contrapositive)** State the contrapositive of the given implication statement in both logical language and English. In both cases, simplify your answer as much as you can by using the De Morgan Law. Please present how you use the De Morgan law in this exercise.

   Implication Statement: "If commodity bundle $x \in \mathbb{R}^N_+$ was chosen when commodity bundle $y \in \mathbb{R}^N_+$ was also affordable at some price $p \in \mathbb{R}^N_{++}$ and income $w > 0$ level, commodity bundle y is never chosen when commodity bundle x is affordable for any price $p' \in \mathbb{R}^N_{++}$ and income level $w' > 0$."[2]

3. **(Geometry)** For three vectors (points) x, y, z in $\mathbb{R}^2$, we suppose they are not collinear. That is, there is no $\lambda \in \mathbb{R}$ such that $\lambda x + (1 - \lambda)y = z$.

   a. Represent the convex hull of set $\{x, y, z\}$ as a set of vectors using two parameters λ_1 and λ_2. State the range of your parameters clearly.

   b. For your choice of vectors, on a plane draw the cone generated by x, y, z:

   $$\text{cone}(x, y, z) = \{\lambda_1 x + \lambda_2 y + \lambda_3 z : \lambda_1, \lambda_2, \lambda_3 > 0\}.$$

   c. Describe mathematically the normal vector of the line passing through point x and point y.

[1] Hint: You need to use the definition for two sets being equal here: $A = B$ when $A \subset B$ and $B \subset A$.

[2] Here, one chooses an bundle x over y whenever $u(x) \geq u(y)$ holds for some utility function $u : \mathbb{R}^N_+ \to \mathbb{R}$. Moreover, a bundle x is affordable at price p and income w means $p \cdot x \leq w$.

d. Describe mathematically the intersection of the half space below the line passing through x, y and the half space below the line passing through y, z. Moreover, for your choice of x, y, z, draw this intersection set.

4. **(Convexity)** Suppose there are $I \geq 2$ agents and $N \geq 1$ goods. Every agent $1 \leq i \leq I$ has a utility function $u_i : \mathbb{R}^N_+ \to \mathbb{R}$.

a. Prove when u_i is concave,[3] the "better than set" $P_i = \{x_i \in \mathbb{R}^N_+ : u_i(x_i) \geq c\}$ is convex for any choice of $c \in \mathbb{R}$.

b. Suppose all better than set P_i is convex. Prove the sum of better than sets $P = \sum_{i=1}^{I} P_i = \{\sum_{i=1}^{I} x_i : x_i \in P_i, \forall i\}$ is convex.

c. Work with two commodities: let $u_i(x) = x_1^{\alpha_1} x_2^{\alpha_2}$ for some $\alpha_1, \alpha_2 > 0$. First, prove u_i is quasi-concave by proving the better than sets it associates is convex. Second, prove u_i is concave if and only if $\alpha_1 + \alpha_2 \leq 1$.

[3] f is concave if $-f$ is convex.