Assignment B Solution

1. **(Budget Set)** When does one set contain another? Give conditions under which

a. $\{x \in \mathbb{R}^2 : p \cdot x \leq b_1\} \subset \{x \in \mathbb{R}^2 : q \cdot x \leq b_2\}$.

b. $\{x \in \mathbb{R}^2_+ : p \cdot x \leq b_1\} \subset \{x \in \mathbb{R}^2_+ : q \cdot x \leq b_2\}$.

where $p, q >> 0$ (all coordinates are positive) and $b_1, b_2 > 0$. Find conditions (on p, q, b_1, b_2) under which these two sets are equal.

Proof. I show for general dimensions. For a, we have two half spaces. So two half space must have the same normal vector. We need $p = \lambda q$ and $b_2 \geq \lambda b_1$ for some $\lambda > 0$. The two sets are equal when $p = \lambda q$ and $b_2 = \lambda b_1$ for some $\lambda > 0$.

For b, the condition is given by $q \leq \lambda p$ and $b_2 \geq \lambda b_1$ for some $\lambda > 0$. It is direct to check this condition implies the set inclusion. To see the necessity of this condition: note the intersection of $\{x : p \cdot x \leq b_1\}$ on the i-th axis is $\frac{b_1}{p_i}$, and the intersection of $\{x : q \cdot x \leq b_2\}$ on the i-th axis is $\frac{b_2}{q_i}$, to have set inclusion, one must have $\frac{b_1}{p_i} \leq \frac{b_2}{q_i}$. Equivalently, $\frac{q_i}{p_i} \leq \frac{b_2}{b_1}$. Define $\lambda = \max_i \frac{q_i}{p_i}$. One has $p \leq \lambda q$ and $b_2 \geq \lambda b_1$.

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2. **(Constraint Qualification)**

a. For the following minimization problem:

$$\min_{x_1, x_2, x_3 \in \mathbb{R}} x_3$$

subject to

$$2x_1 + x_2 = 1$$

$$x_2 = 0$$

$$x_2 + x_3^2 = 0$$

(a1) Find objective function and choice set.[1]

(a2) Find the set of minimizers.

(a3) Define the Lagrangian. (Write down the domain and range of the Lagrangian.)

(a4) Does the constraint qualification condition hold?

(a5) Does the first order condition hold at the minimizer for any choice of the Lagrange multiplier?

[1] Note to define a function, you need to write the domain, the range and the mapping relation of the function.

Solution. For a, the objective function is $f : \mathbb{R}^3 \to \mathbb{R}$ with $f(x_1, x_2, x_3) = x_3$. The choice set is

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : 2x_1 + x_2 = 1, x_2 = 0, x_2 + x_3^2 = 0\} = \{(0.5, 0, 0)\}.$$

Since there is only one feasible choice, $x^* = (0.5, 0, 0)$ is the minimizer. The Lagrangian is defined as $\mathcal{L} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ with

$$\mathcal{L}(x, \lambda) = x_3 + \lambda_1(2x_1 + x_2 - 1) + \lambda_2 x_2 + \lambda_3(x_2 + x_3^2).$$

The constraint qualification condition means the gradient of constraint functions are linearly independent at the minimizer. We have the gradient of the constraints at $(0.5, 0, 0)$ given by $(2, 1, 0)^T$, $(0, 1, 0)^T$ and $(0, 1, 0)^T$. So the constraint qualification condition does not hold. The first order condition at the minimizer $(0.5, 0, 0)$ implies that $\nabla f(x^*) = (0, 0, 1)^T$ is a linear combination of the gradient of constraints $(2, 1, 0)^T$, $(0, 1, 0)^T$ and $(0, 1, 0)^T$, which is never true. Thus, the FOC does not hold at the minimizers. ■

3. **(Slater's Condition)** Consider the following parameterized choice set:

$$\{(x, y, z) \in \mathbb{R}^3_+ : x^2 + y^2 + z^2 \leq \alpha, z = 0\}$$

for some $\alpha \geq 0$. For which values of α do the Slater's condition hold? Justify your answer.

Solution. For b, any $\alpha > 0$ will work by taking $x = y = \sqrt{\frac{\alpha}{8}}$ and $z = 0$. $\alpha = 0$ does not work as no x, y, z such that $x^2 + y^2 + z^2 < 0$. ■

4. **(Linear Programming)** Consider the problem

$$\max_{x \geq 0, y \geq 0} 2x + y$$

subject to

$$x + 3y \leq 19$$

$$x + y \leq 7$$

$$3x + y \leq 11$$

(a) Draw the choice set.
(b) Solve the problem using graph by drawing the level sets of the objective function.
(c) Verify the Slater's condition holds.
(d) Write out the Lagrangian of this problem (Convert it into a minimization problem).

(e) Write out the Karush-Kuhn-Tucker condition.

(f) Solve the problem using the Karush-Kuhn-Tucker method.

Proof. For a,b. See the picture in the end. For c, take $x = y = 0.1$ will do. (note $x = y = 0$ does not as there are five inequality constraint.) For d, the equivalent minimization problem is

$$\min_{x,y} -2x - y$$

subject to

$$x + 3y - 19 \leq 0,$$

$$x + y - 7 \leq 0,$$

$$3x + y - 11 \leq 0,$$

$$-x \leq 0,$$

$$-y \leq 0.$$

the Lagrangian is defined as $\mathcal{L} : \mathbb{R}^2 \times \mathbb{R}^5 \to \mathbb{R}$ with

$$\mathcal{L}(x, y, \lambda) = -2x - y + \lambda_1(x + 3y - 19) + \lambda_2(x + y - 7) + \lambda_3(3x - y - 11) - \lambda_4 x - \lambda_5 y.$$

A pair (x, y, λ) satisfies the KKT condition when

a. Primal feasibility: $x + 3y - 19 \leq 0, x + y - 7 \leq 0, 3x + y - 11 \leq 0, -x \leq 0, -y \leq 0$.

b. Dual feasibility $\lambda_i \geq 0$ for all $i = 1, 2, 3, 4, 5$.

c. Complementary Slackness: $\lambda_1(x+3y-19) = \lambda_2(x+y-7) = \lambda_3(3x-y-11) = \lambda_4 x = \lambda_5 y = 0$.

d. FOC: $\frac{d\mathcal{L}(x,y,\lambda)}{dx} = \frac{d\mathcal{L}(x,y,\lambda)}{dy} = 0$.

Now, we solve the problem. The FOC implies

$$(-2, -1) + (\lambda_1 + \lambda_2 + 3\lambda_3 - \lambda_4, 3\lambda_1 + \lambda_2 - \lambda_3 - \lambda_5) = 0.$$

First, we claim $x > 0$ and $y > 0$ at the maximum. Otherwise, suppose $x = 0$, the problem maximizes y subject to $y \leq \frac{19}{3}, y \leq 7, y \leq 11$. So $y = \frac{19}{3}$. By the complementary slackness, $\lambda_2 = \lambda_3 = \lambda_5 = 0$. So the FOC becomes

$$(-2, -1) + (\lambda_1 - \lambda_4, 3\lambda_1) = 0,$$

which implies $\lambda_1 = -1/3$. Contradicts the dual feasibility. Similarly, suppose $y = 0$. Then, we have the problem maximizes $2x$ subject to $x \leq 19, x \leq 7$ and $x \leq \frac{11}{3}$. So $x = 11/3$. So $\lambda_1 = \lambda_2 = \lambda_4 = 0$. The FOC implies $\lambda_3 < 0$, contradiction to the primal feasibility. Now, suppose $x, y > 0$. So $\lambda_4 = \lambda_5 = 0$. The FOC becomes

$$(-2, -1) + (\lambda_1 + \lambda_2 + 3\lambda_3, 3\lambda_1 + \lambda_2 - \lambda_3) = 0.$$

Note the first three constraints cannot hold as equality at the same time, as the solution of the second and the third constraints at equality, $(2, 5)$, does not make the first constraint an equality. By the complementary slackness, we have $\lambda_1, \lambda_2, \lambda_3$ cannot be positive at the same time. We discuss case-by-case.

case 1 When $\lambda_1 = 0$, $x + 3y < 19$. The FOC implies $\lambda_2 = \frac{5}{4}, \lambda_3 = \frac{1}{4}$. So the second and the third constraints take equality. Thus, $x = 2, y = 5$, which satisfies $x + 3y < 19$. So $(x, y, \lambda) = (2, 5, 0, \frac{5}{4}, \frac{1}{4})$ satisfies the KKT condition.

case 2 When $\lambda_2 = 0$, $x + y < 7$. The FOC implies $\lambda_1 = \lambda_3 = \frac{1}{2}$. So the first and the third constraints take equality. Thus, $x = \frac{7}{4}, y = \frac{23}{4}$. But it violates $x+y < 7$. Contradiction.

case 3 When $\lambda_3 = 0$, $3x + y < 11$. The FOC implies $\lambda_1 = -\frac{1}{2}$ and $\lambda_2 = \frac{5}{2}$. Contradiction with the dual feasibility.

Thus, the maximizer is $(x, y) = (2, 5)$.

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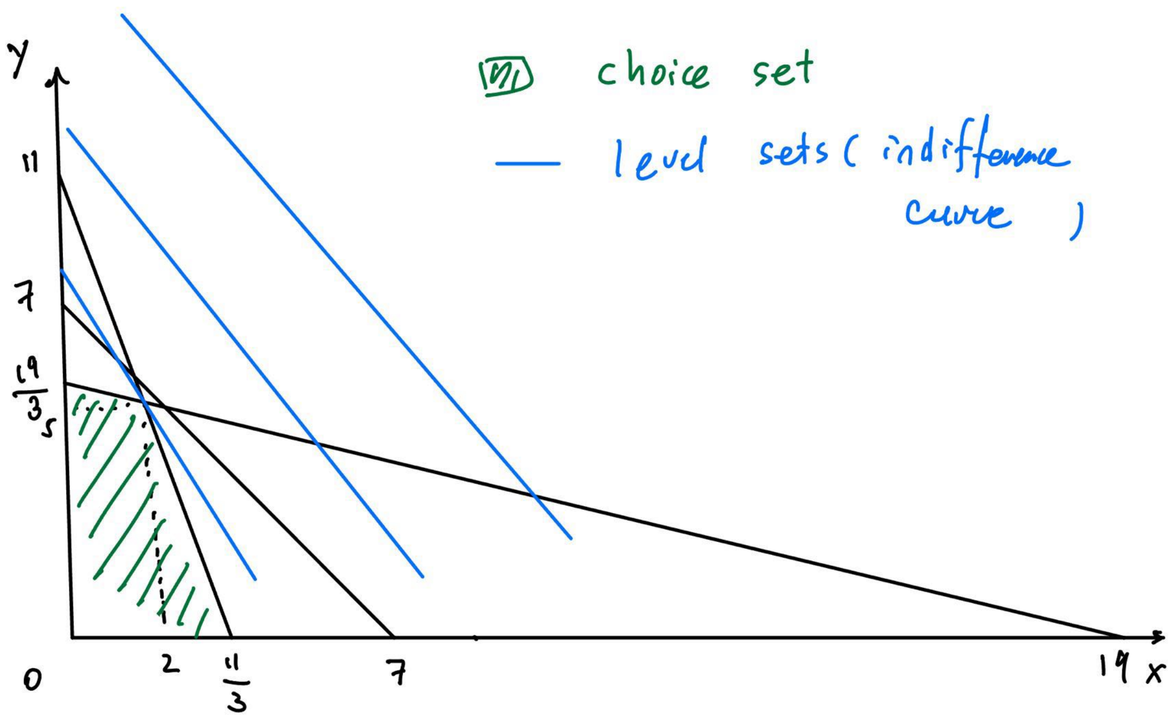

choice set
level sets (indifference curve)
y
x
11
7
19/3
5
0
2
11/3
7
19