

Assignment 1 Solution

Fall 2026.

Advanced Micro.

1(a) We first show $(A \cap B) \cup C \subseteq (A \cup C) \cap (B \cup C)$

Take any $x \in (A \cap B) \cup C$. we have

$$x \in A \cap B \quad \text{or} \quad x \in C$$

if $x \in A \cap B$. we have $x \in A$ and $x \in B$.

so $x \in A \cup C$ and $x \in B \cup C$.

Thus. $x \in (A \cup C) \cap (B \cup C)$

Conversely, we show $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Take any $x \in (A \cup C) \cap (B \cup C)$

we have $x \in A \cup C$ and $x \in B \cup C$

Case 1 $x \in C$. Then $x \in (A \cap B) \cup C$.

Case 2 $x \notin C$. Then $x \in A$ and $x \in B$.

So $x \in A \cap B \Rightarrow x \in (A \cap B) \cup C$.

Since both side inclusion is proved..
we have the set equality.

□

1. (b)

(b1) Suppose y, z , with $y \neq z$, are in $B(p, w)$ and suppose y is chosen.

Then $u(y) > u(z)$. (*)

Suppose z is chosen under (p', w) and y is affordable.

Then it means $y, z \in B(p', w)$ and $u(z) > u(z')$ for any $z' \in B(p', w)$ with $z' \neq z$.

Take $z' = y$. we have $u(z) > u(y)$ (**)

Contradiction between (*) and (**)

(b2). For converse:

If z is not chosen by a consumer under any price-income pair $(p', w) \in \mathbb{R}_{++}^n \times \mathbb{R}$ whenever y is affordable.

Then under some price-income pair $(p, w) \in \mathbb{R}_{++}^n \times \mathbb{R}$ y is chosen over z while both bundles y, z are affordable.

For Contrapositive:

If z is chosen over y when y is affordable under some price-income pair $(p', w) \in \mathbb{R}_{++}^n \times \mathbb{R}$ then y can not be chosen over z when z is affordable under any price-income pair $(p', w) \in \mathbb{R}_{++}^n \times \mathbb{R}$.

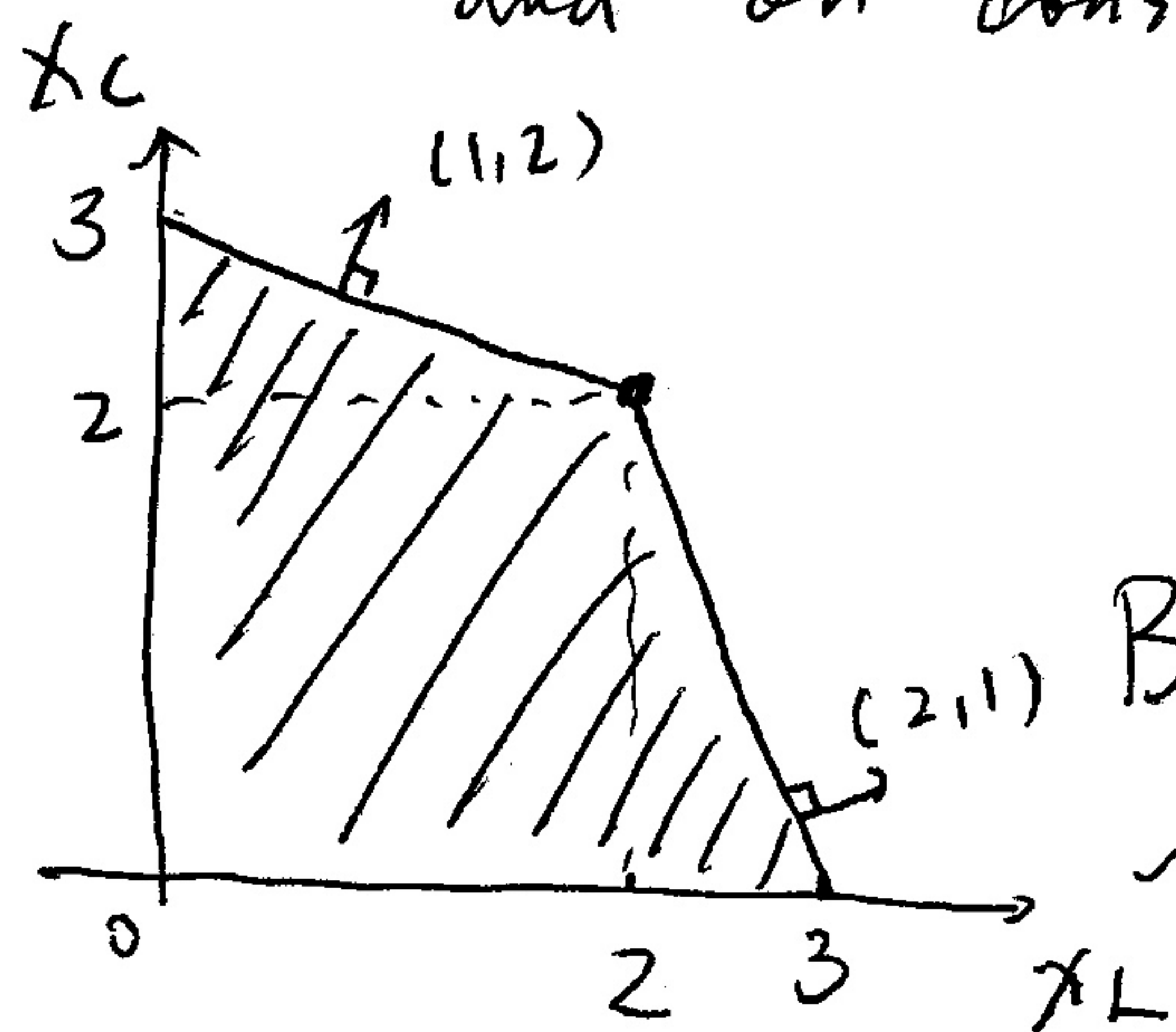
(b3) Consider $u(x_1, \dots, x_n) = x_1 \dots x_n$.

take $x = (1, 0, 0, \dots, 0)$, $y = (0, 1, 0, \dots, 0)$.

Since the utility is Cobb-Douglas, the optimizer is interior. So x is never chosen whenever y is affordable.

But for the same reason, y can't be chosen whenever x is affordable.

2(a1) I denote the consumption quantities of Alice on Leisure as $x_L \geq 0$ and on consumption as $x_C \geq 0$



the budget set of Alice is

$$B = \left\{ (x_L, x_C) \in \mathbb{R}_+^2 : \begin{array}{l} 2x_L + x_C \leq 6 \\ x_L + 2x_C \leq 6 \end{array} \right\}.$$

(a2) Take $x, y \in B \forall \lambda \in [0,1]$. we define

$$z = \lambda x + (1-\lambda)y \quad \text{and show } z \in B.$$

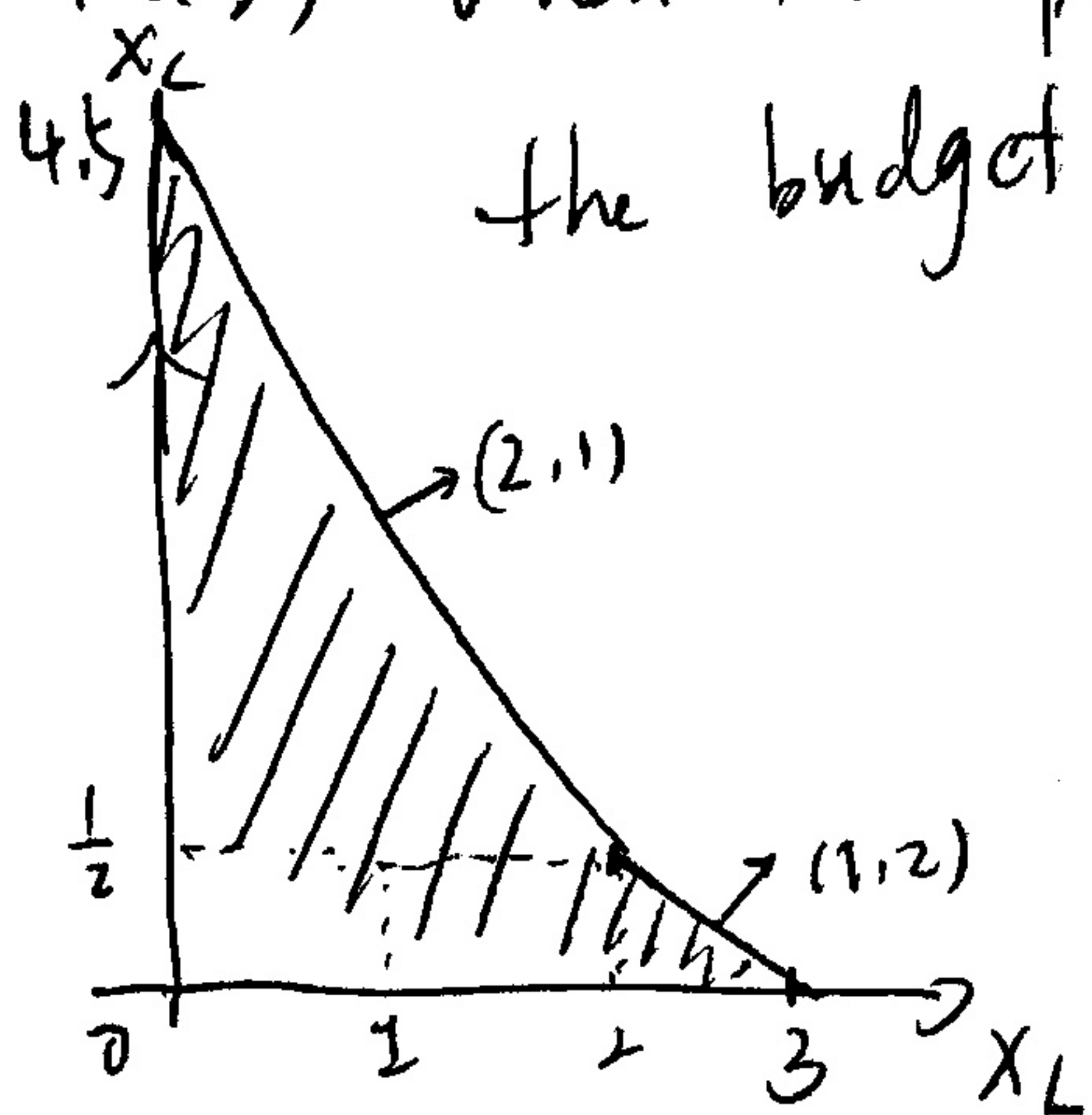
$$\begin{aligned} \text{First, } 2z_L + z_C &= 2(\lambda x_L + (1-\lambda)y_L) + \lambda x_C + (1-\lambda)y_C \\ &= \lambda(2x_L + x_C) + (1-\lambda)(2y_L + y_C) \\ &\leq \lambda \cdot 6 + (1-\lambda) \cdot 6 = 6 \end{aligned}$$

Similarly, we can show $z_L + 2z_C \leq 6$.

Therefore $z \in B$.

That is, B is convex.

1 a3) When the payment scheme becomes progressive.
the budget set will become



$$\tilde{B} = \{ (x_L, x_C) \in \mathbb{R}_+^2 :$$

$$x_L + 2x_C \leq 3$$

$$2x_L + x_C \leq 4.5 \}$$

This set is not convex:

$(3, 0), (0, 4.5)$ are in $\tilde{B}$.

but $\frac{1}{2}[(3, 0) + (0, 4.5)] = (1.5, 2.25) \notin \tilde{B}$

as $1.5 + 2 \cdot 2.25 = 6 > 3$.

□

(b) ~~Pr~~

$$Du(x, y, z) = (y^2 z^3, 2xy z^3, 3xy^2 z^2).$$

So $Du(1, 1, 1) = (1, 2, 3)$, moreover $u(1, 1, 1) = 1$.

and the tangent plane of C at $(1, 1, 1)$ is

$$\{ (x, y, z) \in \mathbb{R}^3 : u(1, 1, 1) + Du(1, 1, 1)[(x, y, z) - (1, 1, 1)] = 1 \}$$

$$= \{ (x, y, z) \in \mathbb{R}^3 : x - 1 + 2(y - 1) + 3(z - 1) = 0 \}$$

$$= \{ (x, y, z) \in \mathbb{R}^3 : x + 2y + 3z = 6 \}.$$

□

3 (a). Alice solves

$$\max_{(x,y) \in \mathbb{R}_+^2} x^2 + y^2$$

$$x+y \leq 1.$$

$$\Leftrightarrow \min_{(x,y) \in \mathbb{R}_+^2} -(x^2 + y^2)$$

$$x+y-1 \leq 0$$

$$-x \leq 0$$

$$-y \leq 0$$

The Lagrangian should be

$$L(x, y, \lambda, v_1, v_2) = -x^2 - y^2 + \lambda(x+y-1) + v_1(-x) + v_2(-y).$$

But since the objective fun is not convex, in the min problem, the KKT theory can't be used.

Bob made a list of mistakes:

- ① the Lagrangian is set up by ignoring the sign of the Lagrangian multiplier.
- ② No multiplier ~~were~~ were assigned to the inequality constraints $x \geq 0$ and $y \geq 0$.
- ③ the problem is with inequality constraints, but was treated as equality constraints and solve by only the FOC over Lagrangian.

④ the objective fun is not convex. so KKT doesn't apply

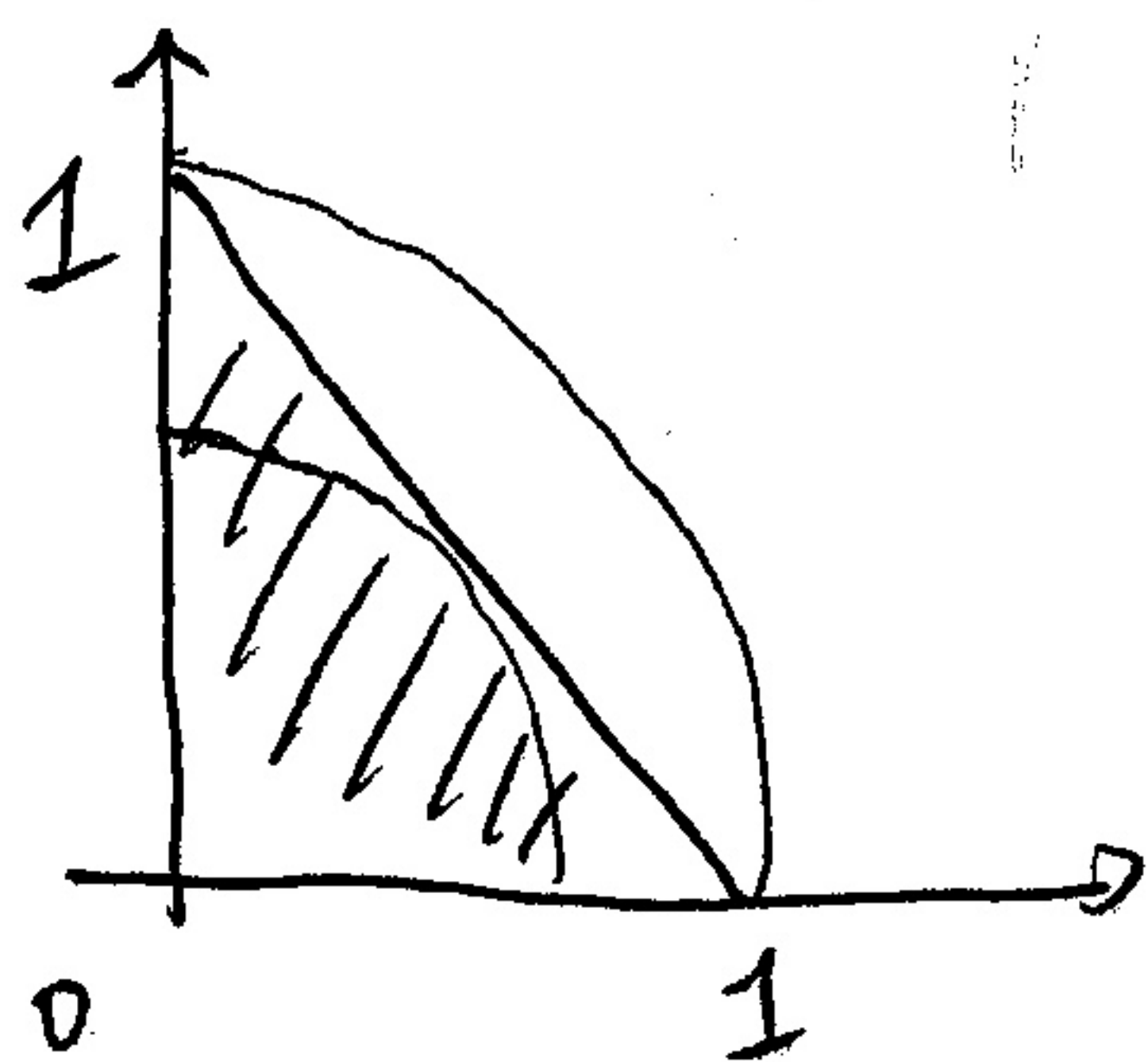
⑤ the FOC is taken over both choice variables and multipliers.

In this form. Bib is solving some version of

$$\min_{(x,y)} x^2 + y^2$$

s.t. $x+y=1$.

(a2) for this problem. I will solve by picture as neither method in class is applicable.



it is straight forward to see that the maximizers are $(1,0)$ and $(0,1)$.

I can even show it formally: for any $x \in B(p,w)$. $x_1^2 + x_2^2 \leq 1$, $x_1 \geq 0$, $x_2 \geq 0$

$$\text{so } x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 \leq 1 - 2x_1x_2 \leq 1.$$

with equality holds when $x = (1,0)$ or $(0,1)$.

(b) The consumer's problem is given by

$$\max_{(x,y) \in B} u(x,y) \Leftrightarrow \max_{x,y} x+y$$
$$x+2y-b \leq 0, 2x+y-b \leq 0$$
$$-x \leq 0, -y \leq 0.$$

Here, the budget set coincides with ^{the form of} (2a), where an economic interpretation is given.

We set up the Lagrangian as

$$\mathcal{L}(x,y,\lambda_1,\lambda_2,\nu_1,\nu_2) = -x-y + \lambda_1(x+2y-b) + \lambda_2(2x+y-b) + \nu_1(-x) + \nu_2(-y)$$

the KKT conditions on $(x,y,\lambda_1,\lambda_2,\nu_1,\nu_2)$ ~~are~~ is:

① Primal feasibility

$$x+2y-b \leq 0, 2x+y-b \leq 0, -x \leq 0, -y \leq 0$$

$$② \text{ Dual feasibility: } \lambda_1, \lambda_2, \nu_1, \nu_2 \geq 0$$

$$③ \text{ Complementary Slackness: } \lambda_1(x+2y-b) =$$
$$\lambda_2(2x+y-b) =$$
$$\nu_1 x = \nu_2 y = 0$$

$$④ \text{ FOC: } -1 + \lambda_1 + 2\lambda_2 - \nu_1 = 0$$

$$-1 + 2\lambda_1 + \lambda_2 - \nu_2 = 0.$$

We need to argue that the Slater's condition holds:

take $(x^*, y^*) = (1, 1)$. we have

$$x^* + 2y^* = 3 < 6 \quad 2x^* + y^* = 3 < 6$$

$$x^* = 1 > 0 \quad y^* = 1 > 0$$

so, the Slater's condition is satisfied.

Moreover, the objective $f(x, y) = -x - y$ is convex, as

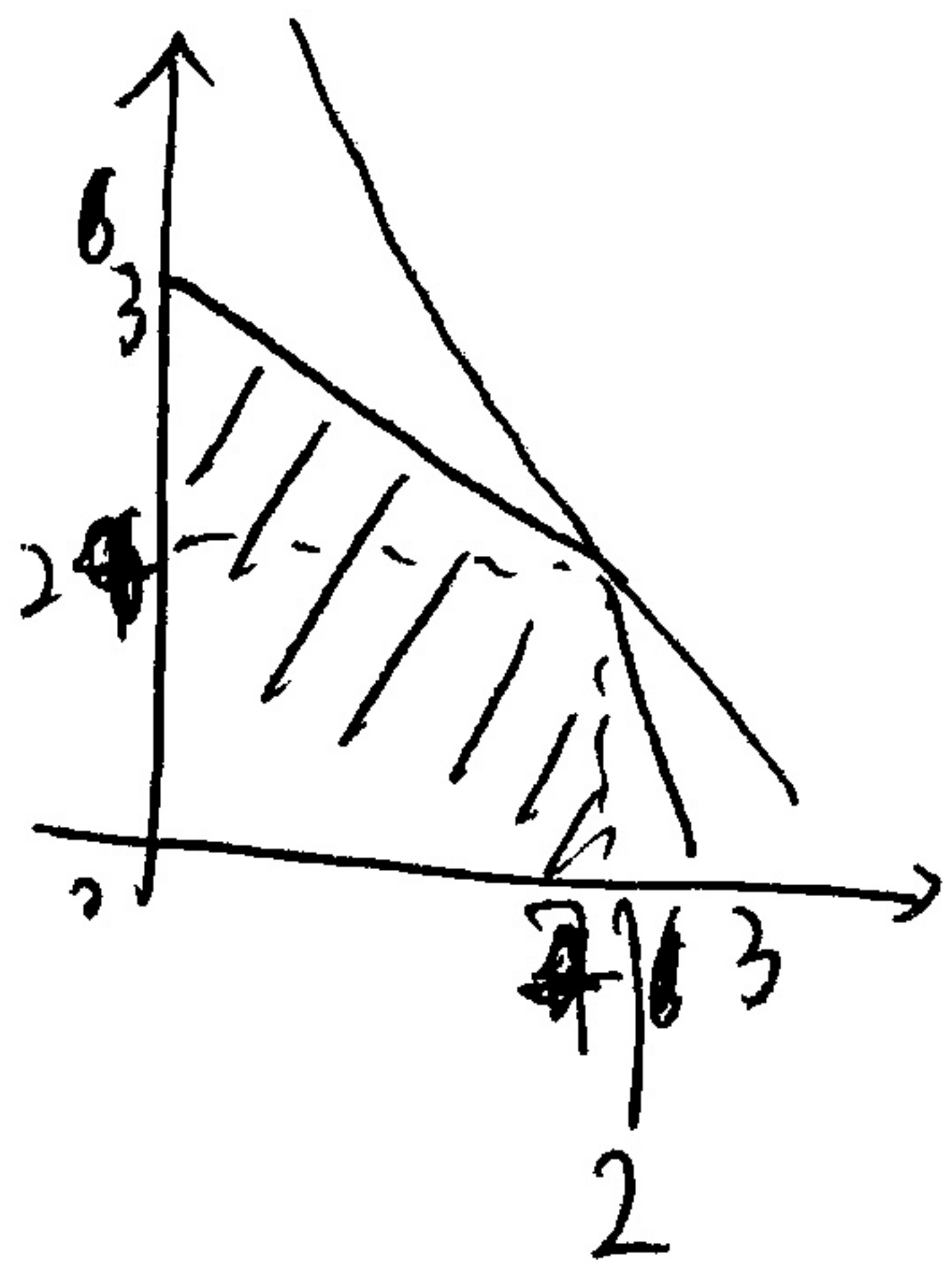
$$H g(x, y) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{everywhere.}$$

So, $H g(x, y)$ is positive semi-definite everywhere

Therefore, g is convex.

So, the KKT condition is necessary and

sufficient for an optimizer.



Now, we show $v_1 = v_2 = 0$.

This is because the picture sol'n tells me (1,1) seems to be a maximiser, with which the inequality constraints $x \geq 0$ and $y \geq 0$ must hold with strict sign.

We prove by contradiction. Suppose ~~not both~~ some v_1 or v_2 is non-zero.

Case 1 $v_1 \neq 0$.

Then, by the complementary slackness, $x = 0$

• it can't be the case that $v_2 \neq 0$ in which case $y = 0$.

which implies $\lambda_1 = \lambda_2 = 0$, by complementary slackness.

Then, FOC gives us $v_1 = v_2 = -1$, contradicting the dual feasibility.

• So $v_2 = 0$.

Since $x = 0$ in this case $\lambda_1(2y - 6) = \lambda_2(y - 6) = 0$

if $y = 3 \Rightarrow \lambda_2 = 0 \Rightarrow \lambda_1 = \frac{1}{2}$ by the second FOC

$\Rightarrow v_1 = \frac{1}{2} - 1 = -\frac{1}{2}$ by the first FOC
Contradiction with dual feasibility

if $y=6 \Rightarrow \cancel{x=0} \Rightarrow \cancel{\lambda_2=1}$
 $x+2y=12 > 6$ violating the primal feasibility

if $y \neq 3$ and $y \neq 6$. $\lambda_1 = \lambda_2 = 0$.

$$\Rightarrow r_1 = \lambda_1 + 2\lambda_2 - 1 = -1 < 0$$

violating the dual feasibility

Thus, it can not be the case $r_1 \neq 0$.

Case 2 $r_2 \neq 0$.

by completely identical argument with indices swapped

between 1 and 2. we get $r_2 \neq 0$ can't hold.

So $r_1 = r_2 = 0$. the FOCs implies $\lambda_1 = \lambda_2 = \frac{1}{3}$.

So by complementary Slackness. $x+2y-6 = 2x+y-6=0$

$$\text{So } x=y=2.$$

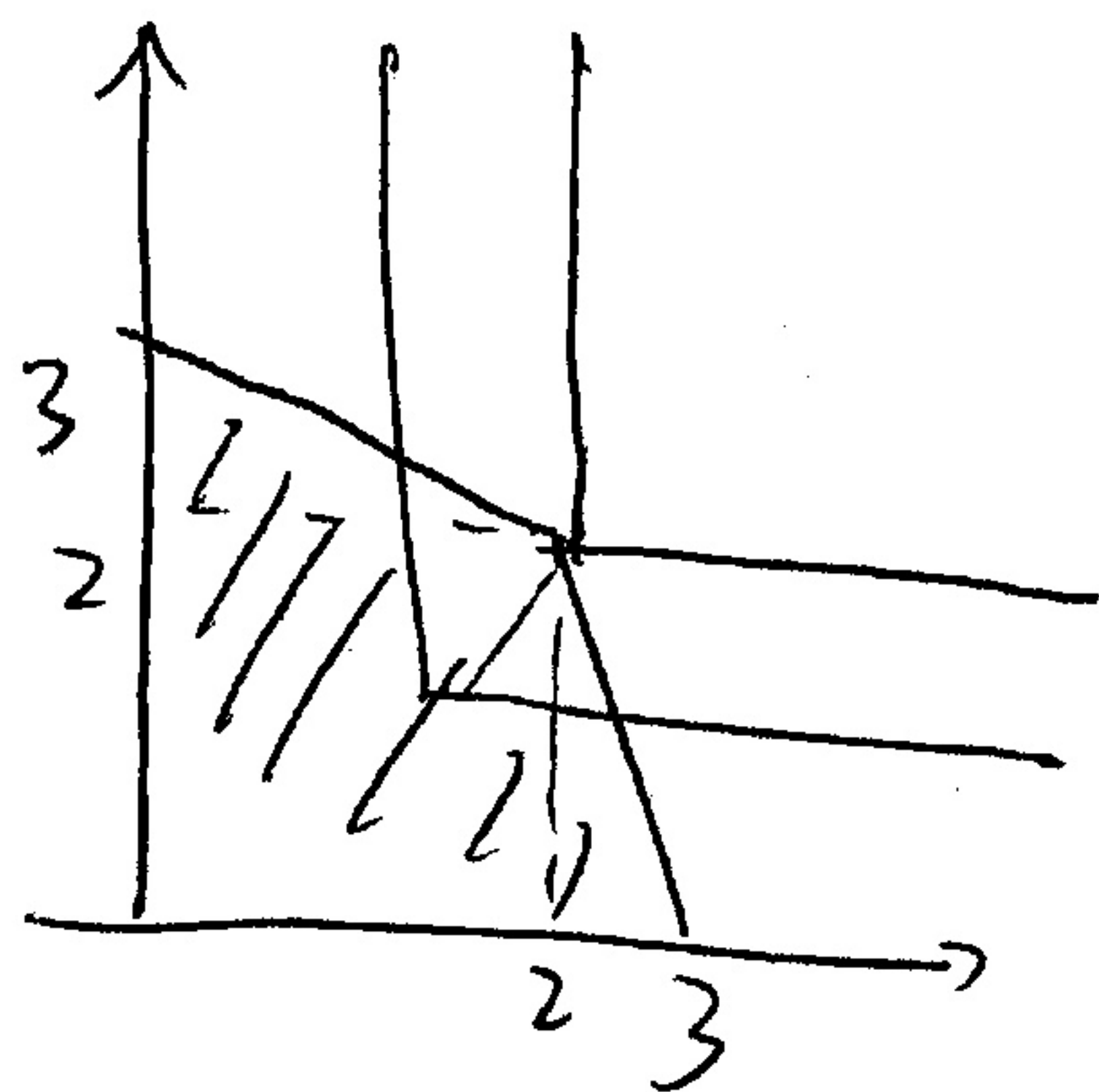
That is. $(x, y, \lambda_1, \lambda_2, r_1, r_2) = (2, 2, \frac{1}{3}, \frac{1}{3}, 0, 0)$

satisfies the KKT condition. and

the solution to the problem is $(2, 2)$.

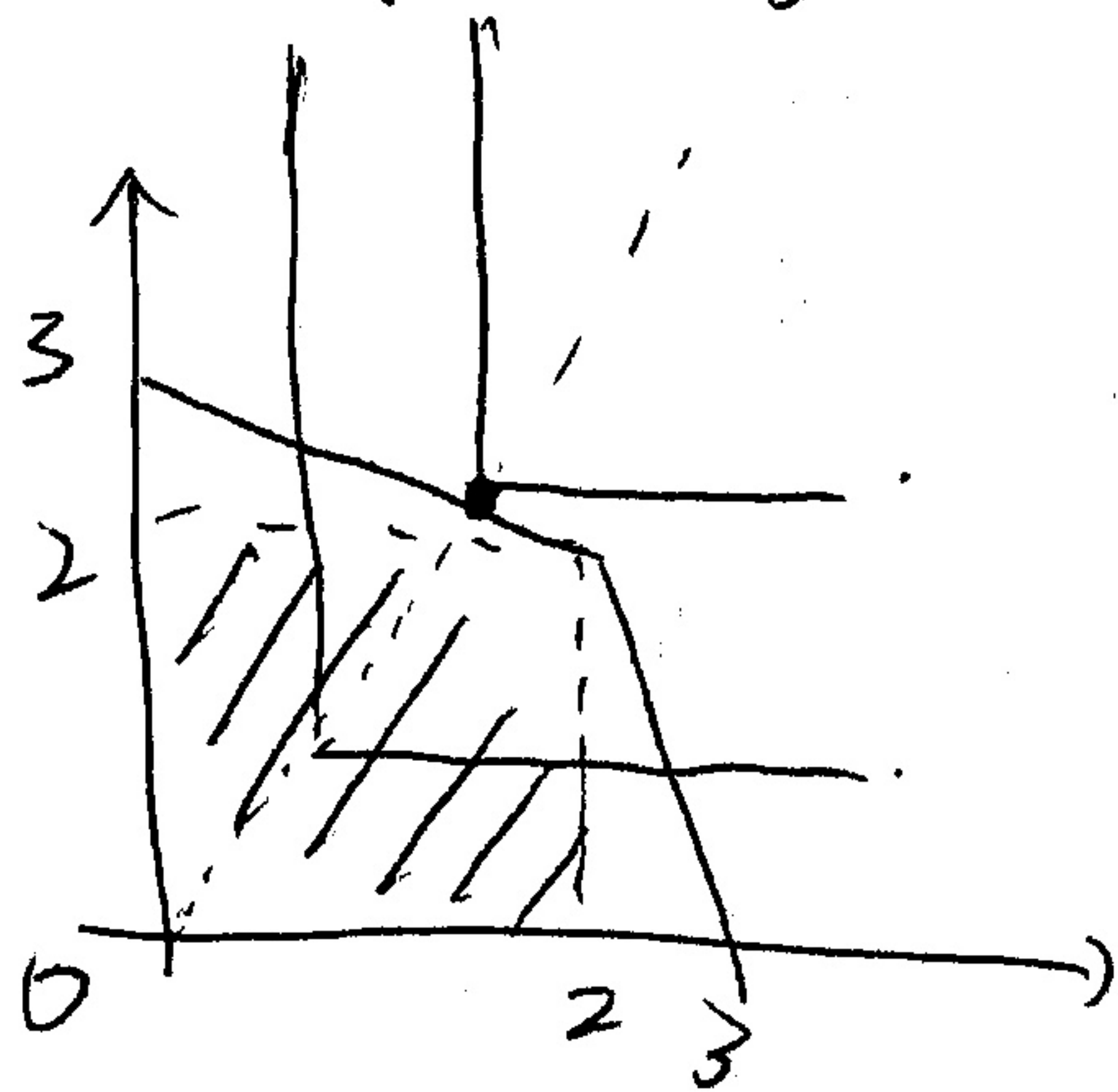
For the second part.

$$\text{if } u(x,y) = \min(x,y).$$



the ~~minimizer~~^{optimizer} remains to be
(2,2)

$$\text{if } u(x,y) = \min(2x,y).$$



the optimizer will no
longer be (2,2) -

4(a). Alice solves

$$V(p, w) = \max_{x, y, z \in \mathbb{R}} xy^2z^3$$

$$(x, y, z) \in B(p, w) = \{(x, y, z) \in \mathbb{R}_+^3 : x + 2y + 3z \leq w\}$$

$$\text{So } -V(p, w) = \min_{(x, y, z) \in B(p, w)} -xy^2z^3$$

with the Lagrangian given by

$$\mathcal{L}(x, y, z, \lambda, \gamma) = -xy^2z^3 + \lambda(x + 2y + 3z - w) + \gamma \cdot (-x, -y, -z).$$

by the envelope thm.

$$-\frac{\partial V(p, w)}{\partial w} = -\lambda(p, w) \Leftrightarrow \frac{\partial V(p, w)}{\partial w} = \lambda(p, w)$$

where $\lambda(p, w)$ solves the consumer's problem's KKT condition.

Since U is Cobb-Douglas, the demand is given by

$$x^*(p, w) = \frac{w}{6p_1} \quad y^*(p, w) = \frac{w}{3p_2} \quad z^*(p, w) = \frac{w}{2p_3}$$

$$\begin{aligned} \text{the FOC gives } \lambda(p, w) &= (y^*(p, w))^2 (z^*(p, w))^3 \\ &= \frac{w^2}{9p_2^2} \cdot \frac{w^3}{8p_3^3} \end{aligned}$$

So the shadow value of money depends on w .

(b). Since u is Cobb-Douglas.

The ^{Marshallian} demand is given by

$$\{p, w\} = \left(\frac{w}{6p_1}, \frac{w}{3p_2}, \frac{w}{2p_3} \right).$$

and the indirect utility function is given by

$$V(p, w) = \frac{w}{6p_1} \cdot \left(\frac{w}{3p_2} \right)^2 \cdot \left(\frac{w}{2p_3} \right)^3 = \frac{w^6}{432 p_1 p_2^2 p_3^3}$$

Using the ~~equivalence~~ duality result.

$$V(p, e(p, v)) = v$$

$$\Rightarrow e(p, v) = (432 p_1 p_2^2 p_3^3 v)^{1/6}$$

Thus, by the Shephard's Lemma.

$$h(p, v) = \nabla_p e(p, v)$$

$$= (432 v)^{1/6} \begin{bmatrix} \frac{1}{6} \cdot p_1^{-5/6} p_2^{1/3} p_3^{1/2} \\ \frac{1}{3} p_1^{1/6} p_2^{-2/3} p_3^{1/2} \\ \frac{1}{2} p_1^{1/6} p_2^{1/3} p_3^{-1/2} \end{bmatrix}$$