# Assignment 2 Solution

Fall 2026

1(a). Note the utility fcn is strictly increasing in both goods. it is locally non-satiated. Therefore. the Walras' Law holds. with

$$p_1 x_1 + p_2 x_2 = w$$

$$\Leftrightarrow \quad x_1 = \frac{w - p_2 x_2}{p_1}$$

This consumer solves

$$\max_{0 \le x_2 \le \frac{w}{p_2}} \quad \frac{w}{p_1} - \frac{p_2}{p_1} x_2 + 2\sqrt{x_2}$$

The $x_2$ - derivative of the objective fcn is

$$-\frac{p_2}{p_1} + x_2^{-1/2}$$

it $\gtrless 0$ iff $x_2 \lessgtr \left(\frac{p_1}{p_2}\right)^2$. Since $x_2 \le \frac{w}{p_2}$.

there are two cases:

Case 1 $\left(\frac{p_1}{p_2}\right)^2 \le \frac{w}{p_2} \Leftrightarrow \underline{w \ge \frac{p_1^2}{p_2}}$.

$$\underline{\xi(p, w) = \left(\frac{w}{p_1} - \frac{p_1}{p_2}, \frac{p_1^2}{p_2^2}\right)}$$

Case 2 $w < \frac{p_1^2}{p_2}$

$$\underline{\xi(p, w) = \left(0, \frac{w}{p_2}\right)}.$$

(b). $V(p,w) = \begin{cases} \frac{w}{p_1} + \frac{p_1}{p_2}, & w \geq \frac{p_1^2}{p_2} \\ 2\sqrt{\frac{w}{p_2}} & w < \frac{p_1^2}{p_2}. \end{cases}$

(c) Using $V(p, e(p,v)) = v$. we have.

· when $e(p,v) \geq \frac{p_1^2}{p_2}$.

$$\frac{e(p,v)}{p_1} + \frac{p_1}{p_2} = v \iff e(p,v) = p_1 v - \frac{p_1^2}{p_2}.$$

In this case.

$$e(p,v) \geq \frac{p_1^2}{p_2} \iff v \geq \frac{2p_1}{p_2}.$$

· when $e(p,v) < \frac{p_1^2}{p_2}$.

$$2\sqrt{\frac{e(p,v)}{p_2}} = v \iff e(p,v) = \frac{p_2 \cdot v^2}{4}$$

In this case.

$$e(p,v) < \frac{p_1^2}{p_2} \iff \frac{p_2 v^2}{4} < \frac{p_1^2}{p_2} \iff v < \frac{2p_1}{p_2}.$$

In sum. $e(p,v) = \begin{cases} p_1 v - \frac{p_1^2}{p_2} & v \geq \frac{2p_1}{p_2} \\ \frac{p_2 v^2}{4} & v < \frac{2p_1}{p_2}. \end{cases}$

(d) by the Shephard Lemma.

$$h(p,v) = D_p e(p,v) = \begin{cases} (v - \frac{2p_1}{p_2}, \frac{p_1^2}{p_2^2}) & v \geq \frac{2p_1}{p_2} \\ (0, \frac{v^2}{4}) & v < \frac{2p_1}{p_2}. \end{cases}$$

1(e). At $p=(2,2)$. $w=10$., $v=V(p,w)=\frac{10}{2}+1=6$.

Since $S(p,w)=D_p h(p,v(p,w))$,

by $v=6>\frac{2p_1}{p_2}$.

$$S(2,2,10)=\begin{pmatrix} -\frac{2}{p_2} & \frac{2p_1}{p_2^2} \\ \frac{2p_1}{p_2^2} & -\frac{2p_1^2}{p_2^3} \end{pmatrix}\Bigg|_{p_1=p_2=2}$$

$$=\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$S(2,2,10)$ is negative semi-definite, because

$\forall x\in\mathbb{R}^2$.

$$x^T\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}x=-x_1^2-x_2^2+2x_1x_2=-(x_1-x_2)^2\le 0.$$

2. (a) by Slutsky's decomposition:

$$\frac{\partial p_i}{\partial p_i}=\frac{\partial h_i}{\partial p_i}-\frac{\partial p_i}{\partial w}p_i$$

Although the Law of demand ensures $\frac{\partial h_i}{\partial p_i}\le 0$.

the commodity $i$ being an inferior good implies

$\frac{\partial p_i}{\partial w}<0$. Therefore, the net effect on

$\frac{\partial p_i}{\partial p_i}$ can be positive, when the income effect is large

(b) because $P_i$ strictly increases while all other prices remain unchanged. we have the ~~budget~~ budget set got strictly ~~larger~~ smaller (in set inclusion). Thus. the utility weakly ~~increases~~ decreases.

To see strict decreases. we have to use $x_i' > 0$ (so in this sense. the framing of the problem is not rigorous.) Note if $x_i = x_i' = 0$ (before and after the price change), no utility change will occur.

If $x_i' > 0$. we have $p \cdot x_i' = p' \cdot x' - (p_i' - p_i) x_i'$

$= w - (p_i' - p_i) x_i' < w.$

Thus. the agent can improve the utility from eating $x_i'$, w/ $u(x_i') = u_i'$, by the local non-satiation.

(c) $CV = e(p', u) - e(p, u)$ satisfying

$V(p', w + CV) = u.$

Note the CV isolates the income effect from the substitution effect. So with $P_i$ becomes relatively more expensive. the compensated demand for good i decreases.

(d) $EV = e(p', u') - e(p, u')$ satisfying

$$V(p, u - EV) = u'.$$

but since i is inferior good. decreasing income increases the consumption of good i.

(e). $$CV = \int_{p_i}^{p_i'} h_i(p, u)\, dp_i$$

$$\overline{EV} = \int_{p_i}^{p_i'} h_i(p, u')\, dp_i$$

Note $h_i(p, u) = g_i(p, e(p, u))$

$h_i(p, u') = g_i(p, e(p, u'))$

with $u' < u$.

Since attaining a lower utility level is strictly less costly. $e(p, u') < e(p, u)$. by the inferiority of good i. $g_i(p, e(p, u)) < g_i(p, e(p, u'))$

i.e. $h_i(p, u) < h_i(p, u')$.

integrating over the changes over $p_i$. we have

$$\underline{CV < EV}.$$

3(a). rationality imposes completeness: $\forall x, y$. $x \succsim y$ or $y \succsim x$.

Take $x = y$. we have $x \succsim x$.

i.e. completeness implies reflexive.

(b) $\succsim$ on $\mathbb{R}^2$ is a preorder (thus a preference) but it is incomplete.

(c) Yes. Take any $x \in \mathbb{R}^2$. we show

$$B = \{ y \in \mathbb{R}^2 : \; y \succsim_L x \} = \{ y \in \mathbb{R}^2 : \; y_1 > x_1 \text{ or } (y_1 = x_1 \text{ and } y_2 \geq x_2) \}$$

is convex.

Take ~~any~~ $a, b \in B$. $\lambda \in [0,1]$. we have

$$a_1 > x_1 \text{ or } (a_1 = x_1 \text{ and } a_2 \geq x_2)$$

and $b_1 > x_1$ or $(b_1 = x_1$ and $b_2 \geq x_2)$.

Take $c = \lambda a + (1-\lambda) b$.

if either $a_1 > x_1$ or $b_1 > x_1$. we have $c_1 > x_1$ so $c \in B$.

if neither, $c_1 = a_1 = b_1 = x_1$. $a_2 \geq x_2$, $b_2 \geq x_2 \Rightarrow c_2 \geq x_2$

so $c \in B$.

i.e. $B$ is convex $\Rightarrow \succsim_L$ is convex.

4. (a1) $u(x,y) = x^{3/7} y^{4/7}$, by the thm in class.

(a2) not a demand. as it violates homogeneity.

(a3) not a demand. as it violates Walras' Law.

(a4) not a demand. as its associated substitution matrix is not negative semi-definite:

by the Slutsky decomposition:

$$S_{11} = \frac{\partial \xi_1}{\partial p_1} + \frac{\partial \xi_1}{\partial w}\xi_1 = \frac{w(p_1^2+p_2^2) - 2p_1^2 w}{(p_1^2+p_2^2)^2} + \frac{p_1^2 w}{(p_1^2+p_2^2)^2}$$

$$= \frac{w p_2^2}{(p_1^2+p_2^2)^2} > 0$$

but a negative semi-definite matrix can not have a positive diagonal term; as $e_1^T S e_1 = S_{11}$. for $e_1 = (1,0)$.

(a5) not a demand. Suppose it is. by the Walras' Law.

$$\forall p, w. \quad a_1 p_1^2 + a_2 p_1 p_2 + p_1 g_1(w) + b_1 p_1 p_2 + b_2 p_2^2 + p_2 g_2(w) = w$$

Differentiate w.r.t. $w$. we have

$$\forall p, w \quad p_1 g_1'(w) + p_2 g_2'(w) = 1.$$

Fix $w$, take $p_1, p_2 \to 0$. we have

LHS $\to 0$ while RHS $\to 1$. Contradiction.

(b1). The income for three days are $w^1 = w^2 = 10$, $w^3 = 9$.

- $p^1 \cdot x^2 = 8 < w^1$, $p^1 \cdot x^3 = 12 > w^1$. $\Rightarrow$ $x^1 \gtrsim_R x^2$, $x^1 \not\gtrsim_R x^3$.
- $p^2 \cdot x^1 = 12 > w^2$, $p^2 \cdot x^3 = 8 < w^2$ $\Rightarrow$ $x^2 \gtrsim_R x^3$, $x^2 \not\gtrsim_R x^1$.
- $p^3 \cdot x^1 = 7 < w^3$, $p^3 \cdot x^2 = 9 = w^3$ $\Rightarrow$ $x^3 \gtrsim_R x^1$, $x^3 \gtrsim_R x^2$.

In this example. WARP doesn't hold. (you can also argue that the revealed preference forms a cyclic relationship).

(b2) We call these data $d^1, \cdots d^5$ in order
$w^1 = 20$. $w^2 = 20$. $w^3 = 22$. $w^4 = 22$. $w^5 = 43$

By $d^1, d^2$. $p^1 \cdot x^2 = 16 < w^1 \Rightarrow x^1 \succ_R x^2$

$p^2 \cdot x^1 = 16 < w^2 \Rightarrow x^2 \succ_R x^1$.

WARP is violated. So $d^1, d^2$ are not from the same person

Similarly. $d^3, d^4$ are not from the same person.

By $d^1, d^4$. $p^1 \cdot x^4 = 14 < w^1 \Rightarrow x^1 \succ_R x^4$

$p^4 \cdot x^1 = 20 < w^4 \Rightarrow x^4 \succ_R x^1$

So. WARP is violated. $d^1, d^4$ are not from the same person

Similarly. $d^2, d^3$ are not from the same person.

Thus. $d^1, d^3$ are from the same person.

$d^2, d^4$ are from the other person.

To determine $d^5$. note

$p^2 \cdot x^5 = 18 < w^2 \Rightarrow x^2 >_R x^5$

$p^5 \cdot x^2 = 36 < w^5 \Rightarrow x^5 >_R x^2$

So WARP is violated. $\Rightarrow$ $d^2, d^5$ are not from the same person

So. $(d^1, d^3, d^5)$ are from one. $(d^2, d^4)$ are from the other one.

Now. we check each data set satisfies GARP.

For $D^1 = (d^1, d^3, d^5)$

$p^1 \cdot x^3 = 16 < w^1 \Rightarrow x^1 >_R x^3$. $p^1 \cdot x^5 = 18 < w^1 \Rightarrow x^1 >_R x^5$

$p^3 \cdot x^1 = 28 > w^3 \Rightarrow x^3 \not>_R x^1$. $p^3 \cdot x^5 = 25 > w^3 \Rightarrow x^3 \not>_R x^5$

$p^5 \cdot x^1 = 48 > w^5 \Rightarrow x^5 \not>_R x^1$. $p^5 \cdot x^3 = 38 < w^5 \Rightarrow x^5 >_R x^3$

There is no revealed preference cycle. So GARP holds.

For $D^2 = (p^2, p^4)$

$p^2 \cdot X^4 = 16 < w^2 \Rightarrow x^2 \succ_R x^4$

$p^4 \cdot x^2 = 28 > w^4 \Rightarrow x^4 \not\succ_R x^2$

So, there is no cycle. GARP is satisfied.