Assignment 3 Solution

Fall 2026

1(a) The budget constraint is

$$B = \{(x_H, x_T) \in \mathbb{R}^2_+ : q_H x_H + q_T x_T \leq w\}$$

Alice's consumption tomorrow is a random variable $c : S \to \mathbb{R}$ with $S = \{H, T\}$ satisfying $c(H) = x_H$, $c(T) = x_T$.

Since only one state occurs, the consumption amount can't be $x_H + x_T$ in general.

(b) the probability (x_H, x_T) is a belief for tomorrow's state. It is not the subjective probability, as Alice ~~does~~ doesn't use expected value to evaluate (unless $u(x) = x$, in which case the probability has two interpretations).

(c) Alice's problem is

$$\max_{(x_H, x_T) \in B} U(x_H, x_T)$$

where $U(x_H, x_T): \mathbb{R}_+^2 \to \mathbb{R}$ is the objective fcn. (x_H, x_T) is the choice variable and B is the choice set.

It can be thought as a classical consumer's problem with each security being a commodity.

The FoC holds when U is quasi-concave and the solution is interior. So when the agent is risk averse and ~~sati~~ the Bernoulli utility fcn u satisfies some Inada condition.

we have $\frac{q_H}{q_T} = \frac{\pi_H u'(x_H)}{\pi_T u'(x_T)}$.

(d) With concave utility, the answer is no.

The course example for the Giffen behavior can not apply, because the utility is in special additive form.

To see ~~lotto~~ the security H is ~~good~~ ~~one~~ normal, we prove by contradiction. Suppose under the new price $q' = (q_H', q_T')$. $x_H' \geq x_H$. By the budget constraint (assuming u is increasing), we have $x_T' \leq x_T$. Thus.

$$\frac{\pi_H u'(x_H)}{q_H'} \leq \frac{\pi_H u'(x_H)}{q_H} = \frac{\pi_T u'(x_T)}{q_T} < \frac{\pi_T u'(x_T')}{q_T}.$$

violating the FOCs.

This argument is not true when u is not concave.

(e) $\mathbb{E}(u(x)) = \pi_H u(x_H) + \pi_T u(x_T)$

$= \pi_H (a x_H - b x_H^2) + \pi_T (a x_T - b x_T^2)$

$= a \mathbb{E}x - b(\pi_H x_H^2 + \pi_T x_T^2)$

$= a \mathbb{E}x - b(\pi_H (x_H - \mu)^2 + \pi_T (x_T - \mu)^2$

$- \mu^2 + 2\mu(\pi_H x_H + \pi_T x_T))$

$= a \mathbb{E}x - b(\mathbb{E}x)^2 - b\,Var(x).$

(f). Now suppose $U(x_H, x_T) = \mathbb{E}x - Var(x)$.

We show U is not vNM. There are 4 axioms:

- complete + transitive: must satisfy as there is a utility representation.
- continuous: satisfy as $x^n \to x$ implies $U(x^n) \to U(x)$.
- independence: Let $U(x) = \frac{1}{2}(x_H + x_T) - \frac{1}{2}(x_H - \frac{x_H + x_T}{2})^2$

$- \frac{1}{2}(x_T - \frac{x_H + x_T}{2})^2$

$= \frac{x_H + x_T}{2} - \frac{(x_H - x_T)^2}{4}$

To see a preference represented by U violates Independence. Let

$X_1 = \$1$ for sure.

$X_2 = \$2$ for sure

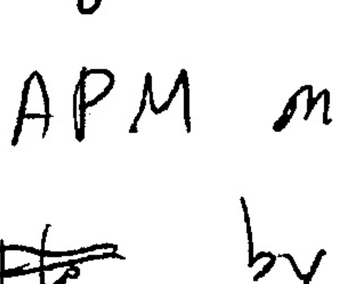
$X_0 = \$0$ for sure.

$U(X_2) > U(X_1)$ but

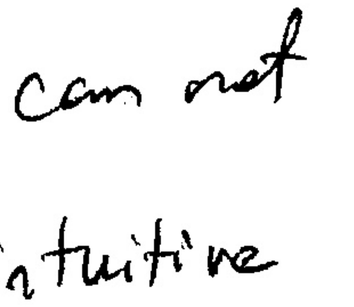
$U(\frac{1}{2}X_2 + \frac{1}{2}X_0) = 1 - 1 = 0 < U(\frac{1}{2}X_1 + \frac{1}{2}X_0) = \frac{1}{4}$.

Comparing e and f, we know that the utility in CAPM models can not always be interpreted ~~as the~~ by the intuitive argument presented in f.

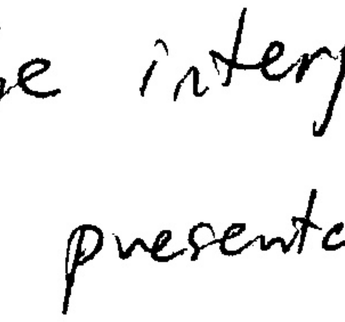
19) Three assets



quantity X_H



X_T



X_B.

First, we suppose $q_B = q_H + q_T$ if there is arbitrage.

today's cash flow $= -q_H x_H - q_T x_T - q_B x_B$

$= -q_H (x_H + x_B) - q_T (x_T + x_B) \geq 0$

So it cannot be the case that both $x_H + x_B \geq 0$ and $x_T + x_B \geq 0$. contradiction. So there is no arbitrage.

Conversely, suppose there is no arbitrage. We prove $q_B = q_H + q_T$ by contradiction. If $q_B > q_H + q_T$.

~~take~~ (the bond price is too high. so one get "free" money buy selling bond today).

take $(x_H, x_T, x_B) = (1+\varepsilon, 1+\varepsilon, -1)$.

todays cash flow $= -(1+\varepsilon)(q_H + q_T) + q_B > 0$
for sufficiently small $\varepsilon > 0$

State - H cash flow $= \varepsilon > 0$

State - T cash flow $= \varepsilon > 0$. so there is arbitrage
Contradiction

if $q_B < q_H + q_T$. (the bond is too cheap,

so one sell securities in exchange of bond)

take $(x_H, x_T, x_B) = (-1, -1, 1+\varepsilon)$

today's cash flow $= 0 \cdot (q_H + q_T) - (1+\varepsilon) q_B > 0$

for sufficiently small $\varepsilon > 0$

State-H cashflow $= \varepsilon > 0$

State-T cashflow $= \varepsilon > 0$

So there is arbitrage. Contradiction.

So $q_B = q_H + q_T \Leftrightarrow$ there is no arbitrage.

3. (a) For u:

$\log(c(x,u)) = \frac{1}{2}\log(x_H) + \frac{1}{2}\log(x_T)$

$\Leftrightarrow \quad c(x,u) = \sqrt{x_H x_T}$

For v:

$(c(x,v) - 3)^3 = \frac{1}{2}(x_H - 3)^3 + \frac{1}{2}(x_T - 3)^3$

$\Leftrightarrow \quad c(x,v) = 3 + \sqrt[3]{\frac{1}{2}(x_H-3)^3 + \frac{1}{2}(x_T-3)^3}$.

1b) $A(w,u) = \frac{-u''(w)}{u'(w)} = \frac{1/w^2}{1/w} = \frac{1}{w}$

$A(w,v) = -\frac{v''(w)}{v'(w)} = \frac{-6(w-3)}{3(w-3)^2} = \frac{2}{3-w}$.

(c) when $w > 3$. $A(w,v) < 0 < A(w,u)$

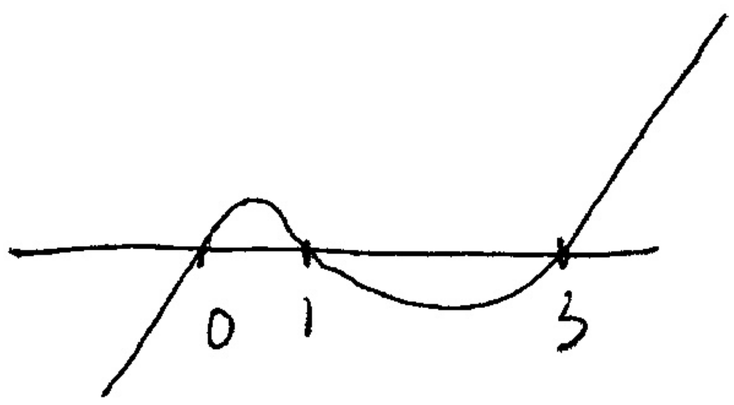
v - agent is risk-loving and

u - agent is risk-averse.

when $0 < w < 3$.

$$\frac{1}{w} \geq \frac{2}{3-w} \iff \frac{3-w-2}{w(3-w)} \geq 0$$

$$\iff \frac{1-w}{w(3-w)} \geq 0$$



$$\iff w(1-w)(3-w) \geq 0$$

$$\iff w(w-1)(w-3) \geq 0$$

$$\iff w \geq 3 \text{ or } 0 \leq w \leq 1.$$

So when $w \in (0,1]$. u - agent is more risk averse

when $w \in (1,3)$. v - agent is more risk averse.

4. (a) Alice solves

$$\max_{0\le\alpha\le w} \frac{1}{2}\sqrt{w-\alpha} + \frac{1}{2}\sqrt{\alpha w}$$

the objective fcn is strictly concave in α.

the FOC is $-\frac{1}{4\sqrt{w-\alpha}} + \frac{\sqrt{w}}{4\sqrt{\alpha}} = 0$

$$\Leftrightarrow \quad \alpha = \frac{w^2}{1+w} \in (0, w)$$

(b) Note $\frac{\partial \alpha}{\partial w} = \frac{w^2+2w}{(1+w)^2} > 0$. the choice strictly increases in w.

(c) If Alice enters. she obtains an expected payoff $\frac{1}{2}\sqrt{w-\frac{w^2}{1+w}} + \frac{1}{2}\sqrt{\frac{w^3}{1+w}}$

$$= \frac{1}{2}\sqrt{\frac{w}{1+w}} + \frac{w}{2}\sqrt{\frac{w}{1+w}}$$

$$= \frac{1}{2}\sqrt{w(1+w)}$$

If Alice does not enter. she obtains $\sqrt{w}$.

entering is better $\Leftrightarrow$ $\frac{1}{2}\sqrt{w(1+w)} \geq \sqrt{w}$

weakly $\Leftrightarrow$ $\frac{1}{2}\sqrt{1+w} \geq 1$

$\Leftrightarrow$ $w \geq 3$.

(d) If $u(z) = z^2$. Alice solves

$$\max_{0 \leq \alpha \leq w} \frac{1}{2}(w-\alpha)^2 + \frac{1}{2}(\alpha w)^2$$

the objective fun is strictly convex in α.

so the maximizer is either $\alpha = 0$ or $\alpha = w$.

when $\alpha = 0$. Alice's expected payoff is $\frac{1}{2}w^2$.

when $\alpha = w$. Alice's expected payoff is $\frac{1}{2}w^4$.

$\frac{1}{2}w^2 \geq \frac{1}{2}w^4 \Leftrightarrow 1 \geq w^2 \Leftrightarrow w \in [0,1]$.

Thus. the optimal investment

$$\alpha = \begin{cases} 0 & w \in [0,1] \\ w & w \in [1,\infty) \end{cases}$$

His investment choice remains the same when $w \in [0,1)$ and increases when $w \in (1, \infty)$.

So entering the bet leads to an expected

payoff $\begin{cases} \frac{1}{2}w^2 & w \in [0,1] \\ \frac{1}{2}w^4 & w \in [1, \infty) \end{cases}$

not entering leads to an expected payoff w^2.

When $w \in [0,1]$, $w^2 \geq \frac{1}{2}w^2$. it is always better to not enter.

When $w \in [1, \infty)$, $\frac{1}{2}w^4 \geq w^2 \Leftarrow w \geq \sqrt{2}$

So entering is weakly better iff $w \geq \sqrt{2}$.