Assignment 1

Advanced Microeconomics I, ITAM, Fall 2021

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1. **Deadline: 4 pm on August 27, 2021**
2. Please submit your work in a PDF file on Canvas and **name your file by the name you preferred to be called**. Unless in very special cases, this deadline will be respected.
3. Collaboration in small groups is encouraged. But each student needs to write his or her own solution independently. By this independence, I mean nobody is supposed to see any other's written solution. In case of any collaboration, please mark your collaborators' names at the beginning of your submitted work.
4. In case any plagiarism is detected, the least severe outcome would be a zero in this entire assignment for all parties that are involved.
5. Please write down your solutions and proofs clearly. In addition, please underline/circle your solution and important steps in your proof.

*Please email me at xinyang.wang@itam.mx for typos or mistakes. Version: August 13, 2021.

1. For statements P, Q, and R, prove the following logical statements:

(1a1) (De Morgen's Law) $\neg(P \wedge Q)$ is logically equivalent to $\neg P \vee \neg Q$.

(1a2) (De Morgen's Law) $\neg(P \vee Q)$ is logically equivalent to $\neg P \wedge \neg Q$.

(1a3) (De Morgen's Law) $\neg(\neg P)$ is logically equivalent to P.

(1b) (Method of Contrapositive) $P \implies Q$ is logically equivalent to $\neg Q \implies \neg P$.

That is, whenever we prove the implication from assumptions implies conclusions, it is equivalent to prove that whenever the conclusion fails, some assumptions must not hold.

(1c) (Method of Contradiction) $\neg(P \implies Q)$ is logically equivalent to $P \wedge \neg Q$.

That is, whenever we prove the implication from assumptions implies conclusions, it is equivalent to first assume both the assumptions are true and the conclusions are wrong, then derive a logical contradiction[1].

(1d) (Transitivity of logical implications) $(P \implies Q) \wedge (Q \implies R) \implies (P \implies R)$.

[1] i.e. a statement that is always false

2.

(2a) Recall $\mathcal{P}(X)$ is the power set of X. Prove $\mathcal{P}(A) = \mathcal{P}(B)$ implies $A = B$.

(2b) For any set X, $\mathcal{F} \subset \mathcal{P}(X)$ is a σ-algebra on X if it satisfies the following properties:

- $\emptyset \in \mathcal{F}$
- For every $A \subset X$, where $A \in \mathcal{F}$, $A^c \in \mathcal{F}$
- For any sequence[2] of subsets $\{A_n\}$ where $A_n \in \mathcal{F}$ for all n, $\bigcup_n A_n \in \mathcal{F}$

2b1 Prove $X \in \mathcal{F}$.

2b2 Prove for any sequence of subsets $\{A_n\}$ where $A_n \in \mathcal{F}$ for all n, $\bigcap_n A_n \in \mathcal{F}$

(Bonus problems continued the next page)

[2] This sequence could be finite or infinite.

(Bonus problems hereafter) Next, we say $\Lambda \subset \mathcal{P}(X)$ is a λ-system if

- $X \in \Lambda$.
- If $A, B \in \Lambda$ and $A \subset B$, then $B - A \in \Lambda$.[3]
- If $\{A_n\}$ is an increasing sequence of subsets, i.e. $A_1 \subset A_2 \subset ...$, with each element being in Λ, then $\bigcup_n A_n \in \Lambda$.

Furthermore, $\Pi \subset P(X)$ is called a π-system, if for any finite sequence $A_1, ..., A_n \in \Pi$, $A_1 \cap ... \cap A_n \in \Pi$. We now assume Π is a π-system such that $\Pi \subset \Lambda$, where Λ is a λ-system. Now, we will try to prove Dynkin's theorem, which states the smallest σ-algebra containing Π is a subset of Λ.

2b3 Let $\lambda(\Pi)$ be the smallest λ-system containing Π.[4] Prove $\lambda(\Pi) \subset \Lambda$.

2b4 Let $B \in \Pi$ and define $\mathcal{A}_B = \{A \subset X : A \cap B \in \lambda(\Pi)\}$. Show $\lambda(\Pi) \subset \mathcal{A}_B$.

2b5 Now let $A \in \lambda(\Pi)$ and define $\mathcal{B}_A = \{B \subset X : A \cap B \in \lambda(\Pi)\}$. Show $\lambda(\Pi) \subset \mathcal{B}_A$.

2b6 Use 2b5 to show $\lambda(\Pi)$ is a π-system.

2b7 So far, we know $\lambda(\Pi)$ is both a π-system and a λ-system, prove it has to be a $\sigma-$algebra.

2b8 Conclude that the smallest σ-algebra containing Π is a subset of Λ.

[3] $B - A = \{x : x \in B, x \notin A\}$

[4] The smallest λ-system containing Π is defined by the intersection of all λ-systems containing Π.

3a. Consider the following matrix

$$A = \begin{pmatrix} 19 & 8 & -2 & 14 & 22 \\ 6 & 3 & -2 & 4 & 7 \\ 7 & 2 & 2 & 6 & 8 \\ 1 & -1 & 4 & 2 & 1 \\ 4 & 5 & -10 & 0 & 5 \end{pmatrix}$$

a1) How many linearly independent columns does this matrix have?

a2) How many linearly independent rows does this matrix have?

a3) Compute the $\dim\{Ax : x \in \mathbb{R}^5\}$.

3b. Show A is positive definite, where

$$A = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix}$$

4a. Let $C \subset \mathbb{R}^n$ be a convex set, with $x_1, ..., x_k \in C$. Let $\lambda_1, ..., \lambda_k \geq 0$ be non-negative numbers such that $\lambda_1 + ... + \lambda_k = 1$. Prove

$$\lambda_1 x_1 + ... + \lambda_k x_k \in C$$

(The definition of convexity is for $k = 2$. We need to prove it in general. Hint: use induction.)

4b. When does one halfspace contain another? Give conditions under which

$$\{x \in \mathbb{R}^n : p \cdot x \leq b_1\} \subset \{x \in \mathbb{R}^n : q \cdot x \leq b_2\}$$

where $p, q \neq 0$. Find conditions under which these two subspaces are equal.

5. Solution set of a quadratic inequality: Let $C \subset \mathbb{R}^n$ be the solution set of a quadratic inequality,

$$C = \{x \in \mathbb{R}^n : x^T A x + b^T x + c \leq 0\}$$

where A is a symmetric $n \times n$ matrix, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

a) Show C is convex if A is positive semi-definite.

(Hint: A set is convex if and only if its intersection with an arbitrary line $\{x + tv : t \in \mathbb{R}\}$ is convex.)

Bonus problem:

b) Show that the intersection of C and the hyperplane defined by $g^T x + h = 0$ (where $g \neq 0$) is convex if $A + \lambda g g^T$ is positive semi-definite for some $\lambda \in \mathbb{R}$.