Assignment 2

Advanced Microeconomics I, ITAM, Fall 2021

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1. **Deadline: 4 pm on September 13, 2021**
2. Please submit your work in a PDF file on Canvas and **name your file by the name you preferred to be called**. Unless in very special cases, this deadline will be respected.
3. Collaboration in small groups is encouraged. But each student needs to write his or her own solution independently. By this independence, I mean nobody is supposed to see any other's written solution. In case of any collaboration, please mark your collaborators' names at the beginning of your submitted work.
4. In case any plagiarism is detected, the least severe outcome would be a zero in this entire assignment for all parties that are involved.
5. Please write down your solutions and proofs clearly. In addition, please underline/circle your solution and important steps in your proof.

*Please email me at xinyang.wang@itam.mx for typos or mistakes. Version: August 31, 2021.

1. (20 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is convex, and let $a, b \in \mathbb{R}$ such that $a < b$,

(a) Show that

$$f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$$

for all $x \in [a, b]$.

(b) Show that

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a} \leq \frac{f(b) - f(x)}{b - x}$$

(c) Suppose f is differentiable. Use the result in (b) to show that

$$f'(a) \leq \frac{f(b) - f(a)}{b - a} \leq f'(b)$$

(d) Suppose f is twice differentiable. Use the result in (c) to show that $f''(a) \geq 0$ and $f''(b) \geq 0$.

2. (20 points)

(a). We say non-empty set $S \subset \mathbb{R}^n$ is *mid-point convex* if for any two points in the sets, their midpoint is in the set. Prove that if a set S is mid-point convex and closed, then S is convex.

(b) A function $f : X \to Y$ is *open* if for all $S \subset X$, $f(S)$ is open. Show that any continuous open function from $\mathbb{R}$ into $\mathbb{R}$ is strictly monotonic. (Hint: an open set in $\mathbb{R}$ should not have a maximum.)

3. (30 points) Let $u : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a bounded function.

(a) Prove that

$$\inf_{x \in \mathbb{R}^n} \sup_{z \in \mathbb{R}^m} u(x, z) \geq \sup_{z \in \mathbb{R}^m} \inf_{x \in \mathbb{R}^n} u(x, z)$$

(b) Call any function $\beta : \mathbb{R}^n \to \mathbb{R}^m$ a strategy for the maximizing side. Denote the set of all such strategies by $\mathcal{B}$. Prove the following identity, and explain why it is not in contrast with (a).

$$\inf_{x \in \mathbb{R}^n} \sup_{z \in \mathbb{R}^m} u(x, z) = \inf_{x \in \mathbb{R}^n} \sup_{\beta \in \mathcal{B}} u(x, \beta(x)) = \sup_{\beta \in \mathcal{B}} \inf_{x \in \mathbb{R}^n} u(x, \beta(x))$$

Now, we say u is *convex-concave* if $u(x, z)$ is a concave function of z, for each fixed x, and a convex function of x, for each fixed z.

(c) Give a second-order condition for a twice differentiable function $u : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ to be convex-concave, by its Hessian $Hu(x, z)$.

(d) Suppose $u : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is convex-concave and differentiable, with $\nabla u(x^*, z^*) = 0$. Prove that the saddle-point property holds: for all x, z, we have

$$u(x^*, z) \leq u(x^*, z^*) \leq u(x, z^*)$$

In addition, prove this implies that u satisfies the strong max-min property:

$$u(x^*, z^*) = \sup_z \inf_x u(x, z) = \inf_x \sup_z u(x, z)$$

(e) Now suppose that $u : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is differentiable, but not necessarily convex-concave, and the saddle-point property holds at (x^*, z^*):

$$u(x^*, z) \leq u(x^*, z^*) \leq u(x, z^*)$$

Prove that $\nabla u(x^*, z^*) = 0$.

4. (30 points) For some vector $c \in \mathbb{R}^n$, an $k \times n$ matrix B, a vector $a \in \mathbb{R}^k$, we study the maximization problem

$$\max_{x \in \mathbb{R}^n_+} c \cdot x$$

$$s.t.\ Bx \leq a.$$

This problem is called the primal problem. The choice variable x is said to be primal feasible if $Bx \leq a$. Suppose that $a >> 0$ and the primal problem has a non-zero solution $\bar{x}$.

a) Use the KKT condition to show that there is a non-negative vector $\bar{y} \in \mathbb{R}^k$ such that $\bar{y} = 0$ if $B\bar{x} < a$, and $\bar{x}$ solves

$$\max_{x \in \mathbb{R}^n_+} [c \cdot x - \bar{y}^T Bx]$$

b) Show that $B^T \bar{y} \geq c$, and $\bar{x}_i = 0$ if $(B^T \bar{y})_i > c_i$.

For the primal problem defined as above, the dual linear program is

$$\min_{y \in \mathbb{R}^k_+} a \cdot y$$

s.t. $B^T y \geq c$

The vector y is dual feasible if $B^T y \geq c$.

c) Show that if x is feasible and y is dual feasible, then

$$c \cdot x \leq y^T Bx \leq a \cdot y$$

d) Show that if $\bar{x}$ solves the primal problem and $\bar{y}$ is as in question (a), then

$$c \cdot \bar{x} = a \cdot \bar{y}$$

e) Show that if $\bar{x}$ solves the primal problem and $\bar{y}$ is as in question (a), then $\bar{y}$ solves the dual problem.

Now, consider a maximization problem

$$\max_{x_1, x_2 \geq 0} 2x_1 + x_2$$

$$s.t.\ x_1 + 3x_2 \leq 19$$

$$x_1 + x_2 \leq 7$$

$$3x_1 + x_2 \leq 11$$

f) Draw the constraint set.

g) Solve this problem.

h) Define the dual problem.

i) Solve the dual problem.