Assignment 3

Advanced Microeconomics I, ITAM, Fall 2021

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1. **Deadline: 11:59 pm on October 8, 2021**
2. Please submit your work in a PDF file on Canvas and name your file by the name you preferred to be called. Unless in very special cases, this deadline will be respected.
3. Collaboration in small groups is encouraged. But each student needs to write his or her own solution independently. By this independence, I mean nobody is supposed to see any other's written solution. In case of any collaboration, please mark your collaborators' names at the beginning of your submitted work.
4. In case any plagiarism is detected, the least severe outcome would be a zero in this entire assignment for all parties that are involved.
5. Please write down your solutions and proofs clearly. In addition, please underline/circle your solution and important steps in your proof.

*Please email me at xinyang.wang@itam.mx for typos or mistakes. Version: September 22, 2021.

1. (10%) A preference $\succeq$ on X is *reflexive* if, for all $x \in X$, $x \succeq x$. While the reflexivity assumption is pretty reasonable, explain why we do not need to invoke this assumption for a rational decision maker.

2. (15%) In each of the following three questions, you are given a data set D of consumption bundles purchased at specific price vectors. In a) and b), either show that the observations could not be the purchases of a single rational consumer, or find a utility function for a rational consumer who makes the purchases. In c), we know the observations in data set D are from two consumers. Partition the observations into two groups such that the observation in each group are consistent with the maximization of a single continuous, quasi-concave utility function.

a) $D = \{(p^i, x^i) : i = 1, 2, 3\}$

- $x^1 = (10, 5, 1), p^1 = (2, 1, 3)$
- $x^2 = (1, 10, 5), p^2 = (3, 2, 1)$
- $x^3 = (5, 1, 10), p^3 = (1, 3, 2)$

b) $D = \{(p^i, x^i), i = 1, 2, 3\}$

- $x^1 = (12, 12, 4), p^1 = (1, 2, 3)$
- $x^2 = (2, 12, 3), p^2 = (3, 1, 2)$
- $x^3 = (9, 12, 18), p^3 = (2, 3, 1)$

c) $D = \{(p^i, x^i), i = 1, 2, 3, 4, 5\}$

- $x^1 = (4, 8), p^1 = (1, 2)$
- $x^2 = (8, 4), p^2 = (2, 1)$
- $x^3 = (4, 6), p^3 = (1, 3)$
- $x^4 = (6, 4), p^4 = (3, 1)$
- $x^5 = (4, 7), p^5 = (2, 5)$

3. (15%) Consider an individual with wealth w who owns a home in a hurricane prone area. A hurricane causes a monetary loss of L dollars if it occurs. The probability of hurricane damage is given by $\alpha \in (0, 1)$. If the homeowner buys $x \geq 0$ units of insurance at a price p per unit, she will receive back x dollars in the event of hurricane damage.

a) What is the expected final wealth of the homeowner if she buys $x \geq 0$ units of insurance at price p.

b) What does it mean here for the price of insurance to be actuarially fair.

c) Suppose the consumer has expected utility preferences with increasing and concave utility u over monetary prizes. Suppose $p = \alpha$. Write down the utility maximization problem for the optimal x and find the solution.

4. (10%) We proved two von Neumann-Morgenstern utility functions represents the same preference if and only if they are positive affine transformations of each other. In its proof, we picked two lotteries p^1 and p^2, and discussed the relative position between a generic lottery p and these two lotteries according to

$$U(p) = \lambda U(p^1) + (1 - \lambda)U(p^2).$$

In lecture note 14, my proof is complete for the case $\lambda \leq 1$, but we missed the part $\lambda > 1$. Please complete the proof by analyzing the case $\lambda > 1$.

5. (20%) For a complete and transitive preference $\succeq$ on $X = \mathbb{R}^n$, prove the following definitions of continuity are equivalent:

1. for any sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}$ in X, if $x_n \to x$, $y_n \to y$ and $x_n \succeq y_n$, we have $x \succeq y$.
2. for $x \succ y$ in X, there exists open balls B_x and B_y around x and y respectively such that $x' \succ y'$ for all $x' \in B_x$, $y' \in B_y$.
3. for any $x \in X$, the better than set $\{y \in X : y \succeq x\}$ and the worse than set $\{y \in X : x \succeq y\}$ are closed.

(Hint: First prove 1 and 2 are equivalent, then use this equivalence to prove 1 and 3 are equivalent.)

6. (10%)

Consider two boxes with the names $ITAM$ and $UNAM$. Each box contains 10000 balls. The balls are either white or black. Box $ITAM$ contains 5001 white balls and 4999 black balls. The percentage of white and black balls in Box $UNAM$ is unknown. The participant of this experiment is asked to choose one box among these two boxes. Once a box is chosen, a ball will be uniformly drawn from this box, and the color of this ball will be announced.

This participant will be asked to choose a box in the following two cases in order.

Case 1: the award is 100 dollars when the drawn ball is white, and 0 otherwise.

Case 2: the award is 100 dollars when the drawn ball is black, and 0 otherwise.

In the experiment, most participants choose Box $ITAM$ in both cases. Now, we analyze this experiment formally.

A *belief* on the color of the chosen ball in box $UNAM$ can be represented by a number $\pi \in [0, 1]$, where π is the probability that the chosen ball is white. Also, we normalize the Bernoulli utility function u defined on $\{0, 100\}$ by $u(0) = 0$, $u(100) = 1$.

Now, we assume that this decision maker holds multiple beliefs. The set of his beliefs is represented by a set $P \subset [0, 1]$. Consider the following utility function over actions $ITAM$ and $UNAM$: For case 1, $U_1 : \{ITAM, UNAM\} \to \mathbb{R}$ is defined by

$$U_1(ITAM) = 0.5001, U_1(UNAM) = \min\{\pi : \pi \in P\}$$

For case 2, $U_2 : \{ITAM, UNAM\} \to \mathbb{R}$ is defined by

$$U_2(ITAM) = 0.4999, U_2(UNAM) = \min\{1 - \pi : \pi \in P\}$$

Namely, his utility from choice $ITAM$ is the expected utility of 100 dollars with the (objective) probability calculated from the distribution of white and black ball in box $ITAM$. However, his utility from choice $UNAM$ is the expected utility of 100 dollars with the probability associated with the most pessimistic belief in P.

a) Prove that if P consists of only one belief, then U_1 and U_2 are derived from a von Neumann-Morgenstern utility function, and $U_1(ITAM) > U_1(UNAM)$ if and only if

$U_2(ITAM) < U_2(UNAM)$.[1]

b) Find a set P for which $U_1(ITAM) > U_1(UNAM)$ and $U_2(ITAM) > U_2(UNAM)$.

[1] i.e. when there is a single belief, a decision maker with a von Neumann-Morgenstern utility function will not choose box ITAM in both cases.

7. (20%)

The choice of information structures must be subject to some limits, otherwise, of course, each agent would simply observe the entire state of the world. - Ken Arrow (1985)[2]

Let X be a finite set of states containing N elements, and $p \in \Delta(X)$ is a probability on X. Given any state $x \in X$, a decision maker could compute his optimal action, which yields him a payoff $u(x) \in \mathbb{R}$.

This decision maker chooses a probability distribution for the states to appear. In an ideal world, he would choose a probability assigning probability 1 on a state yielding him the highest payoff. However, he needs to pay for his choice of probability.

A natural assumption on the cost of choosing probability is that the more uncertain this probability is, the cheaper it should be. A nice measure for the uncertainty would be the Shannon Entropy of a probability: given any $p \in \Delta(X)$,

$$H(p) = -\sum_{x \in X} p(x) \log p(x) \qquad {}^{3}$$

Therefore, a good candidate for the cost of choosing probability p would be $-\lambda H(p)$, where λ could be thought as the unit cost of choosing p.

The decision maker maximizes the net between his expected payoff and his cost on choosing this probability:

$$\max_{p \in \Delta(X)} \sum_{x \in X} p(x)u(x) - \lambda \sum_{x \in X} p(x) \log p(x)$$

(a) Prove

$$\max_{p \in \Delta(X)} \sum_{x \in X} p(x)u(x) - \lambda \sum_{x \in X} p(x) \log p(x) = \lambda \log \left(\sum_{x \in X} \exp \left(\frac{u(x)}{\lambda} \right) \right)$$

(b) Use the technique we studied in class to write the maximization problem as a constrained utility maximization problem. Note the Lagrange multiplier λ should not appear in

[2] Arrow, Kenneth J. "Informational structure of the firm." The American Economic Review 75.2 (1985): 303-307.

[3] The function $H : \Delta(X) \to \mathbb{R}$ is maximized at $p = (1/N, ..., 1/N)$. In addition, at the vertices of the simplex $\Delta(X)$, the function H goes to zero.

your formulation.

(c) Interpret the formulation you proposed in (b).

(Bonus questions hereafter)

The previous problem is a reduced form of a problem on choosing information. More naturally, an information makes a prior more precise via Bayes' rule.

Formally, there are two states in $\Theta = \{\theta_1, \theta_2\}$,[4] and a prior $\pi = (\pi_1, \pi_2) \in \Delta(\Theta)$. There are two signals in $X = \{x_1, x_2\}$.[5] In this case, an information could be formulated by a pair of probabilities $(p_{\theta_1}, p_{\theta_2}) \in \Delta(X)^2$.[6]

(d) Use the prior π and information $(p_{\theta_1}, p_{\theta_2})$ to write

- the probability of observing signal x_1:

  $p(x_1) =$

- the posterior probabilities of states when observing signal x_1:

  $p(\cdot|x_1) = (p(\theta_1|x_1), p(\theta_2|x_1)) =$

(e) Now, we try to redo the previous exercise of formulating a utility maximization problem on choosing information. To achieve this goal, we note

- Given a signal x, the increment of uncertainty could be captured by the non-negative number $H(\pi) - H(p(\cdot|x))$. Following the previous idea, we can talk about the cost of an information. One choice would be the expected increment of uncertainty:

$$\lambda \sum_{x \in X} p(x)(H(\pi) - H(p(\cdot|x))).$$

- Given a state θ and a signal x, we could choose what to do. That is, our payoff would be $u(\theta, x)$.

Formulate a expected utility maximization problem net the cost of information. Please write clearly about the choice set and objective function.

(f) Could you give another formulation of "information"?

[4]You could think of them as a good state and a bad state (for probably whether or not you got a Covid).

[5]You could think of them as a good test result and a bad test result (for a Covid test).

[6]You could think of it as a Covid test. The test is defined by the probabilities of giving a good result when the true state is either good or bad.