Assignment 1

Due: 4pm on September 10, 2025[1]

1. **(Set Operations)** For a price vector $p \in \mathbb{R}^N_{++}$ and a income level $w > 0$, define a budget set as $B(p, w) = \{x \in \mathbb{R}^N_+ : p \cdot x \leq w\}$. Prove, for any $\lambda > 0$, $B(p, w) = B(\lambda p, \lambda w)$.[2]

2. **(Contrapositive)** State the contrapositive of the given implication statement in both logical language and English. In both cases, simplify your answer as much as you can by using the De Morgan Law. Please present how you use the De Morgan law in this exercise.

Implication Statement: "If commodity bundle $x \in \mathbb{R}^N_+$ was chosen when commodity bundle $y \in \mathbb{R}^N_+$ was also affordable at some price $p \in \mathbb{R}^N_{++}$ and income $w > 0$ level, commodity bundle y is never chosen when commodity bundle x is affordable for any price $p' \in \mathbb{R}^N_{++}$ and income level $w' > 0$."[3]

3. **(Convexity)** Suppose there are $I \geq 2$ agents and $N \geq 1$ goods. Every agent $1 \leq i \leq I$ has a utility function $u_i : \mathbb{R}^N_+ \to \mathbb{R}$.

   a. Prove when u_i is concave,[4] the "better than set" $P_i = \{x_i \in \mathbb{R}^N_+ : u_i(x_i) \geq c\}$ is convex for any choice of $c \in \mathbb{R}$.

   b. Suppose all better than set U_i is convex. Prove the sum of better than sets $P = \sum_{i=1}^{I} P_i = \{\sum_{i=1}^{I} x_i : x_i \in P_i, \forall i\}$ is convex.

   c. Now, suppose $u_i(x) = x_1^{\alpha_1} x_2^{\alpha_2} ... x_N^{\alpha_n}$ for some $\alpha_1, ..., \alpha_n > 0$. First, prove u_i is quasi-concave by proving the better than sets it associates is convex. Second, prove u_i is concave if and only if $\alpha_1 + ... + \alpha_n \leq 1$.

4. **(Separating Hyperplane Theorem)** We continue with problem 3. Suppose an allocation $x^* = (x_i^*)_{i \in I} \in (\mathbb{R}^N_+)^I$ is Pareto optimal. Define the better than set for each i as $P_i = \{x_i \in \mathbb{R}^N_+ : u_i(x_i) \geq u_i(x_i^*)\}$, and assume the aggregate translated better than set $\{\sum_{i=1}^{I}(x_i - x_i^*) : x_i \in P_i\}$ is convex. Prove there exists a nonzero vector $p \in \mathbb{R}^N$ such that for every i and every x_i satisfied $u_i(x_i) \geq u_i(x_i^*)$, we have $p \cdot x_i \geq p \cdot x_i^*$.

You will need a stronger version of the theorem here. See Remark 4 of the separating hyperplane theorem in Note 3.

[1] Please submit the physical copy of your work. Write all your statement and deriviations as clearly as you can.

[2] Hint: You need to use the definition for two sets being equal here: $A = B$ when $A \subset B$ and $B \subset A$.

[3] Here, one chooses an bundle x over y whenever $u(x) \geq u(y)$ holds for some utility function $u : \mathbb{R}^N_+ \to \mathbb{R}$. Moreover, a bundle x is affordable at price p and income w means $p \cdot x \leq w$.

[4] f is concave if $-f$ is convex.