Assignment 1 Solution

Due: 4pm on September 10, 2025[1]

1. **(Set Operations)** For a price vector $p \in \mathbb{R}^N_{++}$ and a income level $w > 0$, define a budget set as $B(p, w) = \{x \in \mathbb{R}^N_+ : p \cdot x \leq w\}$. Prove, for any $\lambda > 0$, $B(p, w) = B(\lambda p, \lambda w)$.[2]

Proof. We prove two inclusions.

($\subseteq$) Fix any $x \in B(p, w)$. We have $p \cdot x \leq w$. Multiplying both sides by $\lambda > 0$ gives

$$(\lambda p) \cdot x \;=\; \lambda(p \cdot x) \;\leq\; \lambda w,$$

so $x \in B(\lambda p, \lambda w)$.
($\supseteq$) Fix any $x \in B(\lambda p, \lambda w)$. We have $(\lambda p) \cdot x \leq \lambda w$. Dividing both sides by $\lambda > 0$ yields

$$p \cdot x \;\leq\; w,$$

so $x \in B(p, w)$.

Since $B(p, w) \subseteq B(\lambda p, \lambda w)$ and $B(\lambda p, \lambda w) \subseteq B(p, w)$, we conclude $B(p, w) = B(\lambda p, \lambda w)$. ■

2. **(Contrapositive)** State the contrapositive of the given implication statement in both logical language and English. In both cases, simplify your answer as much as you can by using the De Morgan Law. Please present how you use the De Morgan law in this exercise.

Implication Statement: "If commodity bundle $x \in \mathbb{R}^N_+$ was chosen when commodity bundle $y \in \mathbb{R}^N_+$ was also affordable at some price $p \in \mathbb{R}^N_{++}$ and income $w > 0$ level, commodity bundle y is never chosen when commodity bundle x is affordable for any price $p' \in \mathbb{R}^N_{++}$ and income level $w' > 0$."[3]

Solution. We start by clarifying notations I will use in my solution. You do not need any of these notations (English will work).

1. y is affordable: $y \in B(p, w) = \{x : p \cdot x \leq w\}$.

2. x is chosen under (p, w): $x \in C(p, w)$.

[1] Please submit the physical copy of your work. Write all your statement and deriviations as clearly as you can.

[2] Hint: You need to use the definition for two sets being equal here: $A = B$ when $A \subset B$ and $B \subset A$.

[3] Here, one chooses an bundle x over y whenever $u(x) \geq u(y)$ holds for some utility function $u : \mathbb{R}^N_+ \to \mathbb{R}$. Moreover, a bundle x is affordable at price p and income w means $p \cdot x \leq w$.

The logical formulation of the statement is:

$$\Big(\exists\, p \in \mathbb{R}^N_{++},\ w > 0 : x \in C(p,w) \land\ y \in B(p,w)\Big)$$
$$\Rightarrow \Big(\forall\, p' \in \mathbb{R}^N_{++},\ w' > 0 :\ x \in B(p',w')\ \Rightarrow\ y \notin C(p',w')\Big).$$

The negation of the conclusion is

$$\begin{aligned} &\exists p', w' :\ \neg\big(x \in B(p',w')\ \Rightarrow\ y \notin C(p',w')\big) \\ \iff &\exists p', w' :\ \neg\big(x \notin B(p',w')\ \lor\ y \notin C(p',w')\big) \\ \iff &\exists p', w' :\ \big(x \in B(p',w')\ \land\ y \in C(p',w')\big) \end{aligned}$$

The negation of the assumption is:

$$\begin{aligned} &\forall p, w :\ \neg\big(x \in C(p,w) \land\ y \in B(p,w)\big) \\ \iff &\forall p, w :\ \big(x \notin C(p,w)\ \lor\ y \notin B(p,w)\big) \\ \iff &\forall p, w :\ \big(y \in B(p,w)\ \Rightarrow\ x \notin C(p,w)\big) \end{aligned}$$

Hence, the contrapositive is:

$$\big(\exists p', w' :\ \big(x \in B(p',w')\ \land\ y \in C(p',w')\big)\big) \Rightarrow \big(\forall p, w :\ \big(y \in B(p,w)\ \Rightarrow\ x \notin C(p,w)\big)\big)$$

In English: If there exists some prices and income (p', w') at which x is affordable and y is chosen, then whenever y is affordable at (p, w), x can not be chosen. ■

3. **(Convexity)** Suppose there are $I \geq 2$ agents and $N \geq 1$ goods. Every agent $1 \leq i \leq I$ has a utility function $u_i : \mathbb{R}^N_+ \to \mathbb{R}$.

   a. Prove when u_i is concave,[4] the "better than set" $P_i = \{x_i \in \mathbb{R}^N_+ : u_i(x_i) \geq c\}$ is convex for any choice of $c \in \mathbb{R}$.

   b. Suppose all better than set U_i is convex. Prove the sum of better than sets $P = \sum_{i=1}^I P_i = \{\sum_{i=1}^I x_i : x_i \in P_i, \forall i\}$ is convex.

   c. Now, suppose $u_i(x) = x_1^{\alpha_1} x_2^{\alpha_2} ... x_N^{\alpha_n}$ for some $\alpha_1, ..., \alpha_n > 0$. First, prove u_i is quasi-concave by proving the better than sets it associates is convex. Second, prove u_i is concave if and only if $\alpha_1 + ... + \alpha_n \leq 1$.

[4] f is concave if $-f$ is convex.

Proof. (a) Let $x, y \in P_i(c)$ so $u_i(x) \geq c$ and $u_i(y) \geq c$. For any $\lambda \in [0,1]$, concavity gives

$$u_i\big((1-\lambda)x + \lambda y\big) \;\geq\; (1-\lambda)u_i(x) + \lambda u_i(y) \;\geq\; (1-\lambda)c + \lambda c \;=\; c,$$

so $(1-\lambda)x + \lambda y \in P_i(c)$. Hence $P_i(c)$ is convex.

(b) Let $z = \sum_{i=1}^{I} x_i$ and $z' = \sum_{i=1}^{I} y_i$ with $x_i, y_i \in P_i(c_i)$ for all i. For $\lambda \in [0,1]$,

$$(1-\lambda)z + \lambda z' \;=\; \sum_{i=1}^{I} \big((1-\lambda)x_i + \lambda y_i\big).$$

Because each $P_i(c_i)$ is convex, $(1-\lambda)x_i + \lambda y_i \in P_i(c_i)$ for every i, hence $(1-\lambda)z + \lambda z' \in P$. Therefore P is convex.

(c) Fix a number $c > 0$ ($c \leq 0$ is trivial as the better than set is the whole space).

$$u(x) \geq c \quad \Longleftrightarrow \quad \sum_{k=1}^{N} \alpha_k \log x_k \;\geq\; \log c.$$

The map $g(x) = \sum_k \alpha_k \log x_k$ is concave on $\mathbb{R}^N_{++}$ because it is a sum of concave functions $\alpha_k \log x_k$. Thus the better than sets of u are all convex. That is, u is quasi-concave.

Now, we separately prove the necessity and sufficiency of the exponential requirement for concavity. Define $\alpha = \alpha_1 + ... + \alpha_N$. We compute the Hessian: For $x \in \mathbb{R}^N_{++}$,

$$\frac{\partial u}{\partial x_i} \;=\; \frac{\alpha_i}{x_i} u, \qquad \frac{\partial^2 u}{\partial x_i \partial x_j} \;=\; \begin{cases} \dfrac{\alpha_i \alpha_j}{x_i x_j} u, & i \neq j, \\ \dfrac{\alpha_i(\alpha_i - 1)}{x_i^2} u, & i = j. \end{cases}$$

Let $H(x)$ denote the Hessian at x. For any $z \in \mathbb{R}^N$, set $s_i := z_i / x_i$. Then

$$\begin{aligned} z' H(x) z &= u(x) \left[\sum_{i \neq j} \frac{\alpha_i \alpha_j}{x_i x_j} z_i z_j + \sum_i \frac{\alpha_i(\alpha_i - 1)}{x_i^2} z_i^2 \right] \\ &= u(x) \left[\Big(\sum_i \alpha_i s_i \Big)^2 - \sum_i \alpha_i s_i^2 \right]. \end{aligned} \tag{$*$}$$

Thus the sign of $z' H(x) z$ is the sign of $\big(\sum_i \alpha_i s_i\big)^2 - \sum_i \alpha_i s_i^2$ (since $u(x) > 0$).

To see $\alpha \leq 1 \Rightarrow$ concavity: By the weighted Cauchy–Schwarz inequality,

$$\Big(\sum_i \alpha_i s_i \Big)^2 \;\leq\; \Big(\sum_i \alpha_i \Big) \Big(\sum_i \alpha_i s_i^2 \Big) \;=\; \alpha \sum_i \alpha_i s_i^2.$$

Plugging into $(*)$ gives

$$z'H(x)z \;\le\; u(x)\,\big(\beta-1\big)\sum_i \alpha_i s_i^2 \;\le\; 0 \quad \text{for all } z, x,$$

so $H(x)$ is negative semidefinite and thus u is concave

To see concavity $\Rightarrow$ $\alpha \le 1$: Take any $x \in \mathbb{R}^N_{++}$ and choose $z = x$ (so $s_i \equiv 1$). Then

$$z'H(x)z \;=\; u(x)\,\big(\alpha^2-\alpha\big) \;=\; u(x)\,\alpha(\alpha-1) \;>\; 0,$$

contradicting negative semidefiniteness of $H(x)$. Hence concavity is impossible when $\alpha > 1$. ■

4. **(Separating Hyperplane Theorem)** We continue with problem 3. Suppose an allocation $x^* = (x_i^*)_{i\in I} \in (\mathbb{R}^N_+)^I$ is Pareto optimal. Define the better than set for each i as $P_i = \{x_i \in \mathbb{R}^N_+ : u_i(x_i) \ge u_i(x_i^*)\}$, and assume the aggregate translated better than set $\{\sum_{i=1}^I (x_i - x_i^*) : x_i \in P_i\}$ is convex. Prove there exists a nonzero vector $p \in \mathbb{R}^N$ such that for every i and every x_i satisfied $u_i(x_i) \ge u_i(x_i^*)$, we have $p \cdot x_i \ge p \cdot x_i^*$.

You will need a stronger version of the theorem here. See Remark 4 of the separating hyperplane theorem in Note 3.

Proof. Define set $N = \{0\}$. Note, N is convex, and cannot intersect with the interior of the aggregate translated better than set (let us call A). Mathematically, it means $0 \in A$ but $0 \notin int(A)$. The reason is if $0 \in int_A$, there exists $(x_i)_i$ satisfying $\sum_i x_i = \sum_i x_i^*$ such that everyone is weakly better off and someone is strictly better off. Contradicting the Pareto optimality.

Apply the separating hyperplane theorem to N and A, there exists a $p \neq 0$ such that $p\sum_{i=1}^I (x_i - x_i^*) \ge 0$ whenever all $x_i \in P_i$. Take $x_j = x_j^*$ whenever $j \neq i$. Then, we have $p \cdot x_i \ge p \cdot x_i^*$.

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