Assignment 3

Due: 4pm on October 8, 2025[1]

1. **(Parametrized optimization)** Consider an age-parametrized consumer's problem: there is only one commodity, and the consumption of it is denoted by a number $x \geq 0$. The age parameter a ranges from $(0, 2]$. At age a, the income is a^2, and the happiness is determined by a utility function $u(x, a) = 1 - (x - a)^2$ (that is, this agent is satisfied by consuming a units of the commodity). We denote this agent's optimal happiness level at age a by $V(a)$ and his optimal consumption level by $\mu(a)$. Formally,

$$V(a) = \max_{x \in [0, a^2]} u(x, a)$$

$$\mu(a) = \operatorname*{argmax}_{x \in [0, a^2]} u(x, a)$$

   a. In a coordinate system, draw the graph of functions $u(x, 0.5)$, $u(x, 1)$, $u(x, 1.5)$, $u(x, 2)$.

   b. Highlight points $(\mu(a), V(\mu(a)))$ for $a = 0.5, 1, 1.5, 2$ in your graph in a.

   c. Using a and b, draw the "upper envelope" of this class of optimization problems. That is, draw the curve $(\mu(a), V(\mu(a)))$ for $a \in (0, 2]$.

   d. Compute $V'(1)$ and give an interpretation of this number.[2]

   e. Applying the maximum theorem to prove that μ is a continuous function. Please verify carefully that the assumptions of the maximum theorem holds.

2. **(Maximum Theorem)** In class, we studied the maximum theorem. In particular, we note this theorem assumes: the objective function f is continuous and the choice correspondence φ is continuous. The conclusion is: the value function V is continuous and the maximizer correspondence μ is upper hemi-continuous.

   a. Given an example where f is discontinuous and φ is continuous such that the conclusion of the theorem does not hold.

   b. Given an example where f is continuous, φ is upper hemi-continuous but not continuous such that the conclusion of the theorem does not hold.

   c. Given an example where f is continuous, φ is lower hemi-continuous but not continuous such that the conclusion of the theorem does not hold.

[1] Please submit the physical copy of your work. Write all your statement and deriviations as clearly as you can.

[2] compare to the interpretation of λ as the shadow value of money

3. **(Impossibility of Linear Demand)** Many economists like to work with linear/affine demand. In this exercise, I guide you to prove it is not possible to have a linear/affine demand when there are at least two commodities. By linear/affine demand, I mean the demand function $\xi(p,w) = (\xi_1(p,w), \xi_2(p,w)) \in \mathbb{R}^2_+$ has the form

$$\xi_1(p,w) = a_1 p_1 + a_2 p_2 + g_1(w)$$

$$\xi_2(p,w) = b_1 p_1 + b_2 p_2 + g_2(w)$$

for $a_1, a_2, b_1, b_2 > 0$ and some differentiable functions $g_1, g_2 : \mathbb{R} \to \mathbb{R}$.
The proof goes as follows: we start by assuming there are affine demand. Then,

a. Write out the Walras' law.

b. Differentiate both side of the equation with respect to w. Note the equality you obtained hold for any choice of p and w.

c. Check what happens by fixing $w > 0$ and taking both p_1 and p_2 goes to zero. Find a contradiction by checking the limit on both sides of the equation.

4. **(Lagrange Multiplier)** Given a concave utility function u, for the cost minimization problem

$$e(p,v) = \min_{x \geq 0 : u(x) \geq v} p \cdot x$$

Write out the Lagrangian as a function of the choice variable and the Lagrange multiplier λ on the constraint $u(x) \geq v$.

a. What is the dimension of λ?

b. Argue the cost minimization problem is a convex optimization problem and state when the Slater's condition hold.
Therefore, there is a number λ^* such that the pair $(h(p,v), \lambda^*)$ satisfies the KKT condition.

c. Give an economic interpretation to the Lagrange multiplier at the optimum λ^*.