

Assignment 4

Due: 4pm on October 22, 2025¹

1. **(Consumer's Problem with Quasi-linear Utility Function)** Let $u(x_1, x_2) = x_1 + 2\sqrt{x_2}$ be a utility function for quantities x_1 and x_2 of commodities 1 and 2, respectively. Let p_1, p_2 be the prices of commodities 1 and 2, respectively. Assume $p_1 > 0$ and $p_2 > 0$.

- a. Compute the Marshallian demand $\xi(p_1, p_2, w)$.
- b. Compute the indirect utility function $V(p_1, p_2, w) = \max\{u(x) : p \cdot x \leq w, x \geq 0\}$.
- c. Compute the expenditure function $e(p_1, p_2, v)$.
- d. Compute the Hicksian demand $h(p_1, p_2, v)$.
- e. Compute the substitution matrix $S(p_1, p_2, w)$ at $p_1 = p_2 = 2, w = 10$.
- f. Compute the income effect on good 2, $-\frac{d\xi_2(p_1, p_2, w)}{dw}\xi_2(p_1, p_2, w)$, at $p_1 = p_2 = 2, w = 10$.
- g. (Bonus) Suppose there are two agents with identical utility function u , and denote their consumptions by $x = (x_1, x_2)$ and $y = (y_1, y_2)$ respectively. Suppose the economy has an endowment $\omega = (\omega_1, \omega_2) \gg 0$. We say a pair $(x, y) \geq 0$ is a feasible allocation if $x + y = \omega$. We say a feasible allocation is Pareto optimal if there exists no feasible (x', y') such that either $u(x') > u(x)$ and $u(y') \geq u(y)$ holds or $u(x') \geq u(x)$ and $u(y') > u(y)$ holds.

Prove that a pair (x, y) is Pareto optimal if and only if it solves

$$\max_{(x, y) \text{ feasible}} u(x) + u(y).²$$

2. **(Marshallian Demand)** Each of the following four functions is a possible Marshallian demand function for two commodities at prices p_1 and p_2 , respectively and when wealth is w . In each case, determine whether it is the demand function of a consumer with a locally non-satiated, continuous, and strictly quasi-concave utility function. If it is, say what the utility function is. Otherwise, give a reason.

¹Please submit the physical copy of your work. Write all your statement and derivations as clearly as you can.

²Without quasi-linearity but with some concavity, the best you can get is (x, y) is Pareto optimal if and only if it solves

$$\max_{(x, y) \text{ feasible}} \lambda_1 u(x) + \lambda_2 u(y),$$

for some $\lambda = (\lambda_1, \lambda_2) \geq 0$.

- a. $\xi(p_1, p_2, w) = \left(\frac{wp_2}{2p_1^2}, \frac{wp_1}{2p_2^2} \right).$
- b. $\xi(p_1, p_2, w) = \left(\frac{3}{4} \frac{w}{p_1}, \frac{1}{4} \frac{w}{p_2} \right).$
- c. $\xi(p_1, p_2, w) = \left(\frac{w}{p_1} - \frac{p_2}{p_1^3}, \frac{p_2}{p_1^2} \right).$
- d. $\xi(p_1, p_2, w) = \left(\frac{w\sqrt{p_1}}{p_1^{3/2} + p_2^{3/2}}, \frac{w\sqrt{p_2}}{p_1^{3/2} + p_2^{3/2}} \right).$

3. **(Commodity Type)** A utility function is *homothetic* if

$$u(ax) = au(x) \text{ for all } a > 0$$

- a Prove, when the utility function is homothetic and the Walrasian demand is single-valued, the demand is in the form $\xi(p, w) = g(p)w$ for some function g .³
 - b Prove that if the utility function is homothetic, then there is no Giffen good .
 - c Explain briefly why you would or would not expect utility functions to be homothetic.
4. **(Consumer Welfare)** Consider a price change from the initial price p to a new price p' in which only the price of commodity i decreases. Show if commodity i is inferior, compare the compensating variation (CV) and the equivalence variation (EV).⁴

³Due to this property, homothetic utility functions are very convenient in some applications.

⁴You would need to use and prove that $e(p, v)$ is strictly increasing in v and the equivalence between two types of demand here.