Assignment 4 Solution

Due: 4pm on October 22, 2025[1]

1. **(Consumer's Problem with Quasi-linear Utitlity Function)** Let $u(x_1, x_2) = x_1 + 2\sqrt{x_2}$ be a utility function for quantities x_1 and x_2 of commodities 1 and 2, respectively. Let p_1, p_2 be the prices of commodities 1 and 2, respectively. Assume $p_1 > 0$ and $p_2 > 0$.

   a. Compute the Marshallian demand $\xi(p_1, p_2, w)$.

   b. Compute the indirect utility function $V(p_1, p_2, w) = \max\{u(x) : p \cdot x \leq w, x \geq 0\}$.

   c. Compute the expenditure function $e(p_1, p_2, v)$.

   d. Compute the Hicksian demand $h(p_1, p_2, v)$

   e. Compute the substitution matrix $S(p_1, p_2, w)$ at $p_1 = p_2 = 2$, $w = 10$.

   f. Compute the income effect on good 2, $-\frac{d\xi_2(p_1,p_2,w)}{dw}\xi_2(p_1, p_2, w)$, at $p_1 = p_2 = 2$, $w = 10$.

   g. (Bonus) Suppose there are two agents with identical utility function u, and denote their consumptions by $x = (x_1, x_2)$ and $y = (y_1, y_2)$ respectively. Suppose the economy has an endowment $\omega = (\omega_1, \omega_2) \gg 0$. We say a pair $(x, y) \geq 0$ is a feasible allocation if $x + y = \omega$. We say a feasible allocation is Pareto optimal if there exists no feasible (x', y') such that either $u(x') > u(x)$ and $u(y') \geq u(y)$ holds or $u(x') \geq u(x)$ and $u(y') > u(y)$ holds.

      Prove that a pair (x, y) is Pareto optimal if and only if it solves

$$\max_{(x,y) \text{ feasible}} u(x) + u(y).$$[2]

Solution. For a, note the utility function is locally non-satiated, by the Walras law, we have $x_1 = \frac{w}{p_1} - \frac{p_2}{p_1}x_2$. Substitute it back to the objective function, we have the consumer's problem become

$$\max_{x_2 \geq 0} \left(-\frac{p_2}{p_1}x_2 + 2\sqrt{x_2}\right).$$

[1] Please submit the physical copy of your work. Write all your statement and deriviations as clearly as you can.

[2] Without quasi-linearity but with some concavity, the best you can get is (x, y) is Pareto optimal if and only if it solves

$$\max_{(x,y) \text{ feasible}} \lambda_1 u(x) + \lambda_2 u(y),$$

for some $\lambda = (\lambda_1, \lambda_2) \geq 0$.

By the first order condition, $-\frac{p_2}{p_1} + \frac{1}{\sqrt{x_2}} = 0$, which implies $x_2 = (\frac{p_2}{p_1})^2$ and $x_1 = \frac{w}{p_1} - \frac{p_1}{p_2}$. This solution is valid only when $w \geq \frac{p_1^2}{p_2}$. Otherwise, we have $x_1 = 0$ and $x_2 = \frac{w}{p_2}$. Thus,

$$\xi(p_1, p_2, w) = \begin{cases} (\frac{w}{p_1} - \frac{p_1}{p_2}, \frac{p_1^2}{p_2^2}), & w \geq \frac{p_1^2}{p_2} \\ (0, \frac{w}{p_2}), & w < \frac{p_1^2}{p_2} \end{cases}.$$

For b, simply plug in the demand into the objective function, we have

$$v(p_1, p_2, w) = \begin{cases} \frac{w}{p_1} + \frac{p_1}{p_2}, \frac{p_1^2}{p_2^2}), & w \geq \frac{p_1^2}{p_2} \\ 2\sqrt{\frac{w}{p_2}}, & w < \frac{p_1^2}{p_2} \end{cases}.$$

For c, using $\xi(p, e(p, v)) = h(p, v)$, we have $V(p, e(p, v)) = v$. Therefore,

$$e(p_1, p_2, v) = \begin{cases} vp_1 - \frac{p_1^2}{p_2}, & v \geq \frac{2p_1}{p_2} \\ \frac{p_2 v^2}{4}, & v < \frac{2p_1}{p_2} \end{cases}.$$

For d, using $h(p, v) = D_p e(p, v)$, we have

$$h(p_1, p_2, v) = \begin{cases} (v - \frac{2p_1}{p_2}, \frac{p_1^2}{p_2^2}), & v \geq \frac{2p_1}{p_2} \\ (0, \frac{v^2}{4}), & v < \frac{2p_1}{p_2} \end{cases}.$$

For e, using $S(p, w) = D_p h(p, v(p, w))$, as $w > \frac{p_1^2}{p_2}$, we have

$$S = \begin{pmatrix} -2/p_2 & 2p_1/p_2^2 \\ 2p_1/p_2^2 & -2p_1^2/p_2^2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

For f, we note that $\frac{d\xi_2}{dw} = 0$ when $w \geq \frac{p_1^2}{p_2}$. The income effect on good 2 at (p, w) is defined as $-\frac{d\xi_2}{dw}\xi_2$. Thus, the income effect is zero when $w \geq \frac{p_1^2}{p_2}$.

For g, suppose (x, y) solves the aggregate-utility maximization problem. Then, for any feasible (x', y'), $u(x) + u(y) \leq u(x) + u(y)$, so (x', y') cannot Pareto dominates (x, y). Thus, (x, y) is Pareto optimal.

Conversely, suppose (x, y) is Pareto optimal. We note the both x_2 and y_2 must be interior (Suppose $x_2 = 0$, then transfering a small amount of good 2 from y to x while transfering a lot of good 1 from x to y will be a Pareto improvement). Thus, one must match the marginal rate of substitution:

$$\frac{du(x)/dx_1}{du(x)/dx_2} = \frac{du(y)/dy_1}{du(y)/dy_2}.$$

That is,

$$x_2 = y_2 = 1/2.$$

It is routine to check that any interior feasible $(x_1, \frac{1}{2}), (x_2, \frac{1}{2})$ solves the maximization problem.

■

2. **(Marshallian Demand)** Each of the following four functions is a possible Marshallian demand function for two commodities at prices p_1 and p_2, respectively and when wealth is w. In each case, determine whether it is the demand function of a consumer with a locally non-satiated, continuous, and strictly quasi-concave utility function. If it is, say what the utility function is. Otherwise, give a reason.

a. $\xi(p_1, p_2, w) = \left(\frac{wp_2}{2p_1^2}, \frac{wp_1}{2p_2^2}\right)$.

b. $\xi(p_1, p_2, w) = \left(\frac{3}{4}\frac{w}{p_1}, \frac{1}{4}\frac{w}{p_2}\right)$.

c. $\xi(p_1, p_2, w) = \left(\frac{w}{p_1} - \frac{p_2}{p_1^3}, \frac{p_2}{p_1^2}\right)$.

d. $\xi(p_1, p_2, w) = \left(\frac{w\sqrt{p_1}}{p_1^{3/2} + p_2^{3/2}}, \frac{w\sqrt{p_2}}{p_1^{3/2} + p_2^{3/2}}\right)$.

Solution. The function in a is not a demand as it violates Walras' law. The function in b is a demand as the expenditure is proportional to the income. One utility function is $u(x_1, x_2) = x_1^3 x_2$. The function in c is not a demand as it violates homogeneity. The function in d is not a demand as the substitution matrix is not negative semi-definite: Using $S_{ij} = \frac{d\xi_i}{dp_j} + \frac{d\xi_i}{dw} x_j$, we have

$$S = \frac{w\sqrt{p_1 p_2}}{2(p_1^{3/2} + p_2^{3/2})^2} \cdot \begin{pmatrix} p_2/p_1 & -1 \\ -1 & p_1/p_2 \end{pmatrix}.$$

Take $x = (1, 0)$, we have

$$x^T S x = \frac{w\sqrt{p_1 p_2}}{2(p_1^{3/2} + p_2^{3/2})^2} \cdot \frac{p_2}{p_1} > 0.$$

Thus, S is not negative semi-definite. ■

3. **(Commodity Type)** A utility function is *homothetic* if

$$u(ax) = au(x) \text{ for all } a > 0$$

a Prove, when the utility function is homothetic and the Walrasian demand is single-valued, the demand is in the form $\xi(p, w) = g(p)w$ for some function g. [3]

b Prove that if the utility function is homothetic, then there is no Giffen good .

c Explain briefly why you would or would not expect utility functions to be homothetic.

Proof. For a, by definition, $u(\xi(p, w)) \geq u(x)$ for any x such that $p \cdot x \leq w$. By u is homothetic, we have $u(a\xi(p, w)) \geq u(ax)$ for any $a > 0$. Equivalently, we have $u(a\xi(p, w)) \geq u(y)$ for any y such that $p \cdot x \leq y$. That is, $a\xi(p, w) = \xi(p, aw)$, for all $a > 0$. Hence,

$$\xi(p, w) = w\xi(p, 1) = g(p)w,$$

where $g(p)$ is defined to be $\xi(p, 1)$.

For b, by (a), we note $\frac{d\xi_i}{dw}(p, w) = g_i(p) \geq 0$ for any i, as $\xi(p, 1) \geq 0$. Thus, by the Slutsky decomposition, we have

$$\frac{d\xi_i}{dp_i} = \frac{dh_i}{dp_i} - \xi_i \frac{d\xi_i}{dw} \leq 0,$$

as h_i is downward sloping by the law of demand.

For c, we note in the proof of b, we used the observation that all goods are normal under homothetic utility functions. But I am convinced the existence of inferior good such as low quality cherries. ∎

4. **(Consumer Welfare)** Consider a price change from the initial price p to a new price p' in which only the price of commodity i decreases. Show if commodity i is inferior, compare the compensating variation (CV) and the equivalence variation (EV).[4]

Proof. We show the CV is larger than EV.

First, we show $e(p, v)$ nondecreasing in p: Recall that $e(p, v) = p \cdot h(p, v) \leq p \cdot x$, for any x such that $u(x) \geq v$. Let $p' > p$ in $\mathbb{R}^n_+$, then, we have

$$e(p', v) = p' \cdot h(p', v) \geq p \cdot h(p', v) \geq e(p, v),$$

where the second last inequality is by $p' > p$ and the last inequality is by taking $x = h(p', v)$ in the definition.

Now, we see $e(p, v)$ is strictly increasing in v. We prove by contradiction. Suppose $e(p, v') \leq e(p, v)$ for some $v' > v$, we have $u(h(p, v')) = v' > v$. Thus, shrinking a bit

[3]Due to this property, homothetic utility functions are very convenient in some applications.

[4]First, you would need to use and prove $e(p, v)$ is nondecreasing in p and strictly increasing in v. Then, use the equivalence between two types of demand to proceed.

of consumption will reduce the expenditure: for $\lambda < 1$ but sufficiently close to 1, we have $u(\lambda h(p, v')) > v$. Thus, $e(p, v) \leq p \cdot \lambda h(p, v') < e(p, v')$.

To show the claim, we recall that

$$CV = \int_{p_i'}^{p_i} h_i(p, v) dp_i,$$

$$EV = \int_{p_i'}^{p_i} h_i(p, v') dp_i,$$

To show $CV > EV$, we just need to show $h_i(p, v') < h_i(p, v)$ for all p. Equivalently, we show

$$\xi_i(p, e(p, v')) < \xi_i(p, e(p, v)).$$

Since the new price p' is smaller than the old price p, we know that $v < v'$. Thus, $e(p, v') > e(p, v)$. Since i is inferior, we have $\xi_i(p, e(p, v')) < \xi_i(p, e(p, v))$, and finishes the proof. ■