Assignment 5

Due: 4pm on November 5, 2025[1]

1. **(Preference)**

   (a) Prove if a relation is complete, it must be reflexive. Show by example that a relation that is reflexive can be incomplete.

   (b) Say that a relation $\succeq$ is acyclic if the associated strict relation $\succ$ exhibits no cycles. Show that every transitive relation is acyclic, but show by example that a relation can be acyclic without being transitive.

Proof. For (a), take $x = y$, since xRy or yRx must hold by completeness, we have xRx. The relation $=$ on $\mathbb{R}$ is reflexive but incomplete.

For (b), we first prove that transitivity implies acyclicity. Let $\succeq$ be a transitive relation. Suppose, towards a contradiction, that the associated relation $\succ$ has a cycle: there exist $x_1, ..., x_n$ for some $n > 1$, such that

$$x_1 \succ x_2 \succ ... \succ x_n \succ x_1.$$

Transitivity implies $x_1 \succeq x_n$ and $x_1 \not\preceq x_n$. Contradiction.

We now construct an example of a relation $\succeq$ that is acyclic but not transitive. Take $X = \{x, y, z\}$ and suppose that $\succeq = \{(x, y), (y, z)\}$, with an associated relation such that $x \succ y$ and $y \succ z$. The relation $\succeq$ is acyclic. Now, $\succeq$ is not transitive because $x \succeq y$, $y \succeq z$ but not $x \succeq z$.

One can also argue that the revealed preference relation $\succ_R$ is an example. ■

2. **(Revealed Preference)**

In each of the first two questions, you are given a data set D of consumption bundles purchased at specific price vectors. In each case, first, check if D satisfies the weak axiom of revealed preference. Second, either show D can not be the purchase of a rational consumer or find a utility function for a rational consumer who makes the purchases.

   (a) $D = \{(p^i, x^i) : i = 1, 2, 3\}$

   - $p^1 = (2, 1, 3), x^1 = (3, 1, 1)$
   - $p^2 = (3, 2, 1), x^2 = (1, 3, 1)$

[1] Please submit the physical copy of your work. Write all your statement and deriviations as clearly as you can.

- $p^3 = (1, 2, 2), x^3 = (1, 1, 3)$

(b) $D = \{(p^i, x^i) : i = 1, 2, 3\}$

- $p^1 = (1, 2, 3), x^1 = (3, 3, 1)$
- $p^2 = (3, 2, 1), x^2 = (1, 3, 3)$
- $p^3 = (3, 1, 2), x^3 = (2, 12, 3)$

(c) we know the observations in data set $D = \{(p^i, x^i), i = 1, 2, 3, 4, 5\}$ are from two consumers. Partition the observations into two groups such that the observation in each group are consistent with the maximization of a single continuous, quasi-concave utility function.

- $x^1 = (4, 8), p^1 = (1, 2)$
- $x^2 = (8, 4), p^2 = (2, 1)$
- $x^3 = (4, 6), p^3 = (1, 3)$
- $x^4 = (6, 4), p^4 = (3, 1)$
- $x^5 = (4, 7), p^5 = (2, 5)$

Proof. For (a), similar to the example in Lecture 13, we have $p^1 \cdot x^2 = 8 < p^1 \cdot x^1 = 10$, which implies $x^1 \succ_R x^2$; $p^2 \cdot x^3 = 8 < p^2 \cdot x^2$, which implies $x^2 \succ_R x^3$, and $p^3 \cdot x^1 = 8 < p^3 \cdot x^3 = 10$, which implies $x^3 \succ_R x^1$. Thus, the transitivity is violated and D cannot be rationalized.

(b) The expenditure on commodities 1,2,3 are proportional in the ratio $1 : 2 : 1$. Therefore, the consumer has a Cobb-Douglas utility function, in the form $u(x_1, x_2, x_3) = x_1 x_2^2 x_3$.

(c) We call these data $d^1, ..., d^5$ in order. By d^1, d^2, we have $p^1 \cdot x^2 = 16 < p^1 \cdot x^1 = 20 \Rightarrow x^1 \succ_R x^2$, and $p^2 \cdot x^1 = 16 < p^2 \cdot x^2 = 20 \Rightarrow x^2 \succ_R x_1$, the WARP is violated. Therefore, d^1, d^2 are not from the same person. Similarly, d^3, d^4 are not from the same person.

For d^1, d^4, we have $p^1 \cdot x^4 = 14 < p^1 \cdot x^1 \Rightarrow x^1 \succ_R x^4$, and $p^4 \cdot x^1 = 20 < p^4 \cdot x^4 \Rightarrow x^4 \Rightarrow x^1$. The WARP is violated, which implies d^1, d^4 are not from the same person. Similarly, d^2, d^3 are not from the same person.

Thus, d^1, d^3 are from the same person, and d^2, d^4 are from the other person. To determine d^5, we note $p^2 \cdot x^5 = 18 < p^2 \cdot x^2 \Rightarrow x^2 \succ_R x^5$ and $p^5 \cdot x^2 = 36 < p^5 \cdot x^5 \Rightarrow x^5 \succ_R x^2$. The WARP is violated. Thus d^2, d^5 are not from the same person.

In sum, the necessary condition of WARP suggest d^1, d^3, d^5 are from one person, and d^2, d^4 are from the other person.

To verify that $D^1 = \{d^1, d^3, d^5\}$ and $D^2 = \{d^2, d^4\}$ can be rationalized by a continuous, quasi-concave utility function, we must check D^1, D^2 satisfies the GARP. That is, the revealed preference relation has no cycle.

For D^1, we note $p^1 \cdot x^3 = 16 < p^1 \cdot x^1 \Rightarrow x^1 \succ_R x^3, p^1 \cdot x^5 = 18 < p^1 \cdot x^1 \Rightarrow x^1 \succ_R x^5, p^3 \cdot x^2 = 28 > p^3 \cdot x^3 \Rightarrow x^3 \not\succ_R x^2, p^3 \cdot x^5 = 25 > p^3 \cdot x^3 \Rightarrow x^3 \not\succ_R x^5, p^5 \cdot x^1 = 48 > p^5 \cdot x^5 \Rightarrow x^5 \not\succ_R x^1$, and $p^5 \cdot x^3 = 38 < p^5 \cdot x^5 \Rightarrow x^5 \succ_R x^3$. Therefore, there is no cycle.

For D^2, $p^2 \cdot x^4 = 16 < p^2 \cdot x^2 \Rightarrow x^2 \succ_R x^4$ and $p^4 \cdot x^2 = 28 > p^4 \cdot x^4 \Rightarrow x^4 \not\succ_R x^2$. Therefore, there is no cycle. ■

3. **(Utility Representation)** Write whether the following statements are true or false, and justify your answers:

a. All continuous preferences do not have discontinuous utility representations.

b. Some continuous preference does not have a discontinuous utility representation.

c. All continuous preferences can be represented by a utility function, which has a range given by $[a^*, b^*]$ for $a^* < b^*$.

d. A utility function $u : \mathbb{R}^L_+ \to \mathbb{R}$ satisfies the law of decreasing marginal utility on good i if $\frac{du(x)}{dx_i}$ is decreasing in x_i. Show that increasing transformation of utility functions need not preserve decreasing marginal utility.

Solution. For a, it is false. Consider a preference $\succeq$ on $\mathbb{R}$ with $x \succeq y$ whenever $x \geq y$. It is continuous, but admits a discontinuous utility representation $u : \mathbb{R} \to \mathbb{R}$ with $u(x) = x$ for $x \geq 0$ and $u(x) = x - 1$ for $x < 0$.

For b, it is true. Suppose the preference is a constant preference: for all $x, y \in X$, $x \sim y$. Then, the only utility representation must be a constant utility representation. It admits no discontinuous utility (which necessarily implies some alternative is better than some other).

For c, by Debreu's theorem, there exists a continuous utility representation $u : X \to \mathbb{R}$. The range of u must be an nontrivial interval $[a, b]$ given the preference is nontrivial. In particular, we allow a, b to be $-\infty$ or ∞. Next, we can simply argue that there exists an increasing transformation that transform this interval $[a, b]$ to the target $[a^*, b^*]$. There are four cases.

Case 1 When all a, b, a^*, b^* are finite, this can be done by a positive affine transformation whenever $a \neq b$: take $\Phi : \mathbb{R} \to \mathbb{R}$ be $\Phi(x) = a^* - \frac{a(b^*-a^*)}{b-a} + \frac{b^*-a^*}{b-a}x$, we have $\Phi([a, b]) = [a^*, b^*]$. When $a = b$, the preference is trivial, simply represents it to any number in $[a^*, b^*]$ will do the work.

Case 2 When $a = -\infty$ and $b < \infty$, take $\Phi : \mathbb{R} \to \mathbb{R}$ be $\Phi(x) = a^* + \frac{b^*-a^*}{e^b} e^x$, we have $\Phi([a,b]) = [a^*, b^*]$. So $\Phi(u)$ is a utility representation satisfies the requirement.

Case 3 When $a > -\infty$ and $b = \infty$, take $\Phi : \mathbb{R} \to \mathbb{R}$ be $\Phi(x) = b^* + \frac{a^*-b^*}{e^{-a}} e^{-x}$, we have $\Phi([a,b]) = [a^*, b^*]$, and Φ is increasing. So $\Phi(u)$ is a utility representation satisfies the requirement.

Case 4 When $a = -\infty$ and $b = \infty$, take $\Phi : \mathbb{R} \to \mathbb{R}$ be $\Phi(x) = \frac{a^*+b^*}{2} + \frac{b^*-a^*}{\pi} \arctan(x)$, we have $\Phi([a,b]) = [a^*, b^*]$, and Φ is increasing. So $\Phi(u)$ is a utility representation satisfies the requirement.

Note, here I showed only the range of the utility presentation can be designed. One can show that the image of utility presentation can be designed in the same manner.

For d., consider $u(x) = \sqrt{x}$ in a 1-commodity space. Let $f = x^4$. Then, $f(u(x)) = x^2$ has increasing marginal utility. ■

4. **(Debreu's Theorem)** There are two equivalent definitions for the continuity of preference $\succeq$ on a set X:

Def 1 for any sequence x_n, y_n in set X, $x_n \succeq y_n$ for all $n \in \mathbb{N}$ implies $x \succeq y$.

Def 2 All the better than sets and worse than sets are closed. Formally, for all $x \in X$, $B(x) = \{y \in X : y \succeq x\}$ and $W(x) = \{y \in X : x \succeq y\}$ are closed.

A binary relation $\succeq$ on $\mathbb{R}^2_+$ defines a preference as follows: for any pairs of vectors $(x,y), (x',y') \in \mathbb{R}^2_+$,

- if $x \neq y$ and $x' \neq y'$, $(x,y) \succeq (x',y')$ if and only if $xy \geq x'y'$;
- if $x = y$ but $x' \neq y'$, $(x,y) \succeq (x',y')$ if and only if $xy \geq x'y' - 1$;
- if $x \neq y$ but $x' = y'$, $(x,y) \succeq (x',y')$ if and only if $xy \geq x'y' + 1$;
- if $x = y$ and $x' = y'$, $(x,y) \succeq (x',y')$ if and only if $x \geq x'$.

Answer the following questions:

(a) Is $\succeq$ defined above rational?

(b) Use Def 1 to check if $\succeq$ is continuous.

(c) Use Def 2 to check if $\succeq$ is continuous.

(d) Find a utility representation of $\succeq$.[2]

(e) Is $\succeq$ convex?

Proof. For (a), for every $(x, y), (x', y') \in \mathbb{R}^2_+$, if $x \neq y$ and $x' \neq y'$, then either $xy \geq x'y'$ or $x'y' \geq xy$. Therefore, the two vectors are comparable under $\succeq$. When $x = y$ but $x' \neq y'$, then either $xy \geq x'y' - 1$ or $x'y' > xy + 1 \geq xy - 1$. Therefore, two vectors are comparable. When $x = y$ and $x' = y'$, then either $x \geq x'$ or $x' \geq x$. The two vectors are comparable. Therefore, $\succeq$ is complete.

For transitivity, suppose $(x, y) \succeq (x', y')$ and $(x', y') \succeq (x'', y'')$. We discuss by cases:

- If $x = y$,
  - if $x' = y'$, then $x \geq x'$. If $x'' = y''$, then $x' \geq x''$, which implies $x \geq x'$, which implies $(x, y) \succeq (x'', y'')$. If $x'' \neq y''$, then $x'y' \geq x''y'' - 1$. But $x \geq x'$ so $xy = x^2 \geq (x')^2 = x'y'$. Therefore, $xy \geq x''y'' - 1$. Therefore, $(x, y) \succeq (x'', y'')$.
  - If $x' \neq y'$. Then, $xy \geq x'y' - 1$. If $x'' = y''$, then $x'y' \geq x''y'' + 1$, which implies $xy \geq x''y''$, and $(x, y) \succeq (x'', y'')$. If $x'' \neq y''$, we have $x'y' \geq x''y''$. Therefore, $xy \geq x''y'' - 1$, which implies $(x, y) \geq (x'', y'')$.
- If $x \neq y$,
  - if $x' = y'$, then $x \geq x'$. If $x'' = y''$, then $xy \geq x'y' + 1$. If $x'' \neq y''$, then $x'y' \geq x''y'' - 1$. Then, $xy \geq x''y''$, which implies $(x, y) \succeq (x'', y'')$. If $x'' = y''$, we have $x' \geq x''$. Therefore, $xy \geq x''y'' + 1$, which implies $(x, y) \succeq (x'', y'')$.
  - If $x' \neq y'$. Then, $xy \geq x'y'$. If $x'' = y''$, then $x'y' \geq x''y'' + 1$, which implies $xy \geq x''y'' + 1$, and $(x, y) \succeq (x'', y'')$. If $x'' \neq y''$, we have $x'y' \geq x''y''$. Therefore, $xy \geq x''y''$, which implies $(x, y) \geq (x'', y'')$.

Hence, $\succeq$ is transitive. Thus it is rational.

For (b), take $v^n = (1.5, 1)$ and $w^n = (1 + 1/n, 1 - 1/n)$. Each $v^n \succeq y^n$ because $1.5 \cdot 1 > 1 - 1/n^2$. However, in the limit $(1.5, 1) \not\succeq (1, 1)$. Thus, $\succeq$ is not continuous.

For (c), consider the lower contour set $W(x)$ of $v = (1.5, 1)$. Each $w^n = (1+1/n, 1-1/n)$ is the set, but not its limit. Thus, $W(x)$ is not closed. Thus, $\succeq$ is not continuous.

For (d), the utility function is given by

$$u(x, y) = \begin{cases} xy & x \neq y \\ xy + 1 & x = y \end{cases}.$$

[2]If your answer in either b or c is $\succeq$ is discontinuous, it means a discontinuous preference could have a utility representation (though maybe discontinuous).

For (e), take $v = (1.5, 1)$ and $w = (1, 1)$ with $w \succeq v$ by definition. But $\frac{v+w}{2} = (1.25, 1) \not\succeq v$. That is, $\succeq$ is not convex. ■