Assignment 6

Due: 4pm on November 19, 2025[1]

1. (**Uniqueness of vNM**) We proved two von Neumann-Morgenstern utility functions represents the same preference if and only if they are positive affine transformations of each other. In its proof, we picked two lotteries p^1 and p^2, and discussed the relative position between a generic lottery p and these two lotteries according to

$$U(p) = \lambda U(p^1) + (1 - \lambda)U(p^2).$$

In lecture note 14, my proof is complete for the case $\lambda \leq 1$, but we missed the part $\lambda > 1$. Please complete the proof by analyzing the case $\lambda > 1$.

2. (**Expected Utility**) For a non-singleton outcome set $C \subset \mathbb{R}$, check if the following function is von Neumann-Morgenstern utility function:

$$U(p) = \mathbb{E}(p) - Var(p).$$

Here, p is a lottery on C, the expectation $\mathbb{E}(p) = \sum_{c \in C} p(c)c$ and the variance $Var(p) = \sum_{c \in C} p(c)(c - \mathbb{E}(p))^2$. If you think it is, prove it by finding the Bernoulli utility function. If you think it is not, prove your claim.

3. **(Risk Attitude)** Consider two agents who use von Neumann-Morgenstern utility function to evaluate uncertainties. Their preferences are parametrized by two Bernoulli utility functions on $\mathbb{R}_+$:

$$u(w) = \log w$$

$$v(w) = (w - 1)^3$$

For the uncertainty, suppose only the case that all uncertainties are generated by a coin flip. That is, there are two states - Head (H) and Tail (T), each of which happens with probability 0.5. In this situation, a random variable is denoted by two real numbers $x(H), x(T)$.

a. Derive the certainty equivalence for a random variable $x = (x(H), x(T))$ for both agents.

b. Write down the Arrow-Pratt index for both agents.

c. Could you try to compare the risk attitude of two agents?

[1]Please submit the physical copy of your work. Write all your statement and deriviations as clearly as you can.

4. **(Ambiguity)** In this example, we try to give an explanation of a seemingly paradoxical experiment by assuming a decision maker may have multiple beliefs.

Consider two boxes A and B. Each box contains 100 balls. The balls are either white or black. Box A contains 51 white balls and 49 black balls. The percentage of white and black balls in Box B is unknown. The participant of this experiment is asked to choose one box among these two boxes. Once a box is chosen, a ball will be uniformly drawn from this box, and the color of this ball will be announced.

This participant will be asked to choose a box in the following two cases in order.

Case 1: the award is 100 dollars when the drawn ball is white, and 0 otherwise.

Case 2: the award is 100 dollars when the drawn ball is black, and 0 otherwise.

In the experiment, most participants choose Box A in both cases. Now, we analyze this experiment formally.

A *belief* on the color of the chosen ball in box B can be represented by a number $\pi \in [0, 1]$, where π is the probability that the chosen ball is white. Also, we normalize the Bernoulli utility function u defined on $\{0, 100\}$ by $u(0)$=0,u(100) = 1.

Now, we assume that this decision maker holds multiple beliefs. The set of his beliefs is represented by a set $P \subset [0, 1]$. Consider the following utility function over actions A and B: For case 1, $U_1 : \{A, B\} \to \mathbb{R}$ is defined by

$$U_1(A) = 0.51, U_1(B) = \min\{\pi : \pi \in P\}$$

For case 2, $U_2 : \{A, B\} \to \mathbb{R}$ is defined by

$$U_2(A) = 0.49, U_2(B) = \min\{1 - \pi : \pi \in P\}$$

Namely, his utility from choice A is the expected utility of 100 dollars with the (objective) probability calculated from the distribution of white and black ball in box A. However, his utility from choice B is the expected utility of 100 dollars with the probability associated with the most pessimistic belief in P.

a. Prove that if P consists of only one belief, then U_1 and U_2 are derived from a von Neumann-Morgenstern utility function, and $U_1(A) > U_1(B)$ if and only if $U_2(A) < U_2(B)$.[2]

b. Find a set P for which $U_1(A) > U_1(B)$ and $U_2(A) > U_2(B)$.

[2] i.e. when there is a single belief, a decision maker with a von Neumann-Morgenstern utility function will not choose box 1 in both cases.