Assignment 6 Solution

Due: 4pm on November 19, 2025[1]

1. (**Uniqueness of vNM**) We proved two von Neumann-Morgenstern utility functions represents the same preference if and only if they are positive affine transformations of each other. In its proof, we picked two lotteries p^1 and p^2, and discussed the relative position between a generic lottery p and these two lotteries according to

$$U(p) = \lambda U(p^1) + (1-\lambda)U(p^2).$$

In lecture note 14, my proof is complete for the case $\lambda \leq 1$, but we missed the part $\lambda > 1$. Please complete the proof by analyzing the case $\lambda > 1$.

Proof. We pick p^1 and p^2 such that $U(p^1) < U(p^2)$. We swish to show that $U(p) = aV(p) + b$, where p satisfies $U(p) = \lambda U(p^1) + (1-\lambda)U(p^2)$ for some $\lambda > 1$. Therefore, we have

$$U(p^1) = \frac{1}{\lambda}U(p) + (1 - \frac{1}{\lambda})U(p^2).$$

By using the linearity of U, with $1/\lambda \in (0,1)$, we have

$$U(p^1) = U(\frac{1}{\lambda}p + \frac{\lambda-1}{\lambda}p^2).$$

As U, V represent the same preference, we have

$$V(p^1) = V(\frac{1}{\lambda}p + \frac{\lambda-1}{\lambda}p^2) = \lambda V(p) + \frac{\lambda-1}{\lambda}V(p^2).$$

Therefore, $V(p) = \lambda V(p^1) + (1-\lambda)V(p^2)$. Next, the procedure is the same as in the notes:

$$V(p) = V(p^2) - \lambda(V(p^2) - V(p^1)) = \frac{V(p^2) - V(p^1)}{U(p^2) - U(p^1)}U(p) + V(p^2) - \frac{U(p^2)(V(p^2) - V(p^1))}{U(p^2) - U(p^1)}.$$

Take $a = \frac{V(p^2)-V(p^1)}{U(p^2)-U(p^1)} > 0$ and $b = V(p^2) - \frac{U(p^2)(V(p^2)-V(p^1))}{U(p^2)-U(p^1)}$, we have $V(p) = aU(p) + b$. Here the choice of a and b are the same as in the cases that $\lambda \leq 1$. ■

2. (**Expected Utility**) For a non-singleton outcome set $C \subset \mathbb{R}$, check if the following function is von Neumann-Morgenstern utility function:

$$U(p) = \mathbb{E}(p) - Var(p).$$

[1] Please submit the physical copy of your work. Write all your statement and deriviations as clearly as you can.

Here, p is a lottery on C, the expectation $\mathbb{E}(p) = \sum_{c\in C} p(c)c$ and the variance $Var(p) = \sum_{c\in C} p(c)(c - \mathbb{E}(p))^2$. If you think it is, prove it by finding the Bernoulli utility function. If you think it is not, prove your claim.

Proof. To see U is not a von Neumann-Morgenstern utility function, we prove it violates independence. Let p^1 be the lottery having a value 2 with a probability 1/2 and having a value 0 with a probability 1/2, p^2 be the deterministic lottery having a value 0 with a probability 1, p^3 be the deterministic lottery having a value 1 with a probability 1.

Thus, $U(p^1) = 1 - 1 = 0 = U(p^2)$. Therefore, $p^1 \sim p^2$.

Let $\lambda = 1/2$. We have $\lambda p^1 + (1-\lambda)p^3$ is a lottery having a value 2 with a probability 1/4, having a value 1 with a probability 1/2, having a value 0 with a probability 1/4. Therefore, $\mathbb{E}(\lambda p^1 + (1-\lambda)p^3) = 1$ and $Var(\lambda p^1 + (1-\lambda)p^3) = \frac{1}{2}$. Thus, $U(\lambda p^1 + (1-\lambda)p^3) = \frac{1}{2}$.

Meanwhile, $\lambda p^2 + (1-\lambda)p^3$ is a lottery having a value 1 with a probability 1/2, having a value 0 with a probability 1/2. Therefore, $\mathbb{E}(\lambda p^2 + (1-\lambda)p^3) = 1/2$ and $Var(\lambda p^2 + (1-\lambda)p^3) = \frac{1}{4}$. Thus, $U(\lambda p^2 + (1-\lambda)p^3) = \frac{1}{4} < U(\lambda p^1 + (1-\lambda)p^3)$.

Therefore, $\lambda p^1 + (1-\lambda)p^3 \succ \lambda p^2 + (1-\lambda)p^3$, violating the independence axiom. Thus, U is not vNM. ∎

3. **(Risk Attitude)** Consider two agents who use von Neumann-Morgenstern utility function to evaluate uncertainties. Their preferences are parametrized by two Bernoulli utility functions on $\mathbb{R}_+$:

$$u(w) = \log w$$

$$v(w) = (w-1)^3$$

For the uncertainty, suppose only the case that all uncertainties are generated by a coin flip. That is, there are two states - Head (H) and Tail (T), each of which happens with probability 0.5. In this situation, a random variable is denoted by two real numbers $x(H), x(T)$.

a. Derive the certainty equivalence for a random variable $x = (x(H), x(T))$ for both agents.

b. Write down the Arrow-Pratt index for both agents.

c. Could you try to compare the risk attitude of two agents?

Proof. For a, the certainty equivalence of x is defined by a deterministic random variable with outcome $c(x)$ such that $c(x) \sim x$. When the Bernoulli utility function is $u(x)$,

$$\log(c(x)) = 0.5\log(x(H)) + 0.5\log(x(T)) \iff c(x) = \sqrt{x(H)x(T)}.$$

When the Bernoulli utility function is $v(x)$,

$$(c(x)-1)^3 = 0.5(x(H)-1)^3+0.5(x(t)-1)^3 \iff c(x) = 1+\sqrt[3]{0.5(x(H)-1)^3+0.5(x(t)-1)^3}.$$

For b, the Arrow-Pratt index is defined by $A_u(w) = -\frac{u''(w)}{u'(w)}$. Thus, for $u(w) = \log(w)$, $A_u(w) = \frac{1}{w}$. For $v(w) = (w-1)^3$, $A_v(w) = \frac{2}{1-w}$. Note $v'(1) = 0$ but $v''(1) > 0$. Therefore, the Arrow-Pratt index of v is undefined when $w = 1$. Another way to see its nonexistence the limit from two sides of 1 are different for $A_v(1)$.

For c, we compare the Arrow-Pratt indices. When $w < 1$, we have

$$A_u(w) > A_v(w) \iff \frac{1}{w} > \frac{2}{1-w} \iff w < 1/3.$$

Therefore, on $(0, \frac{1}{3})$, the log agent is more risk averse; on $(\frac{1}{3}, 1)$, the cubic agent is more risk averse. When $w > 1$, the cubic agent is risk loving with $A_v(w) > 0$. Thus the risk averse log agent is more risk averse than the cubic agent.

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4. **(Ambiguity)** In this example, we try to give an explanation of a seemingly paradoxical experiment by assuming a decision maker may have multiple beliefs.

Consider two boxes A and B. Each box contains 100 balls. The balls are either white or black. Box A contains 51 white balls and 49 black balls. The percentage of white and black balls in Box B is unknown. The participant of this experiment is asked to choose one box among these two boxes. Once a box is chosen, a ball will be uniformly drawn from this box, and the color of this ball will be announced.

This participant will be asked to choose a box in the following two cases in order.

Case 1: the award is 100 dollars when the drawn ball is white, and 0 otherwise.

Case 2: the award is 100 dollars when the drawn ball is black, and 0 otherwise.

In the experiment, most participants choose Box A in both cases. Now, we analyze this experiment formally.

A *belief* on the color of the chosen ball in box B can be represented by a number $\pi \in [0, 1]$, where π is the probability that the chosen ball is white. Also, we normalize the Bernoulli utility function u defined on $\{0, 100\}$ by $u(0)$=0,u(100) $= 1$.

Now, we assume that this decision maker holds multiple beliefs. The set of his beliefs is represented by a set $P \subset [0, 1]$. Consider the following utility function over actions A and B: For case 1, $U_1 : \{A, B\} \to \mathbb{R}$ is defined by

$$U_1(A) = 0.51, U_1(B) = \min\{\pi : \pi \in P\}$$

For case 2, $U_2 : \{A, B\} \to \mathbb{R}$ is defined by

$$U_2(A) = 0.49, U_2(B) = \min\{1 - \pi : \pi \in P\}$$

Namely, his utility from choice A is the expected utility of 100 dollars with the (objective) probability calculated from the distribution of white and black ball in box A. However, his utility from choice B is the expected utility of 100 dollars with the probability associated with the most pessimistic belief in P.

a. Prove that if P consists of only one belief, then U_1 and U_2 are derived from a von Neumann-Morgenstern utility function, and $U_1(A) > U_1(B)$ if and only if $U_2(A) < U_2(B)$.[2]

b. Find a set P for which $U_1(A) > U_1(B)$ and $U_2(A) > U_2(B)$.

Proof. For a, if $P = \{\pi\}$, we have $U_1(B) = \pi$ and $U_2(B) = 1 - \pi$. Hence, both U_1 and U_2 are derived from the expected utility

$$pu(100) + (1 - p)u(0).$$

When deriving U_1, p is the probability that the drawn ball is white, and when deriving U_2, p is the probability that the drawn ball is black.

Moreover, we have

$$U_1(A) > U_1(B) \iff 0.51 > \pi \iff 0.49 < 1 - \pi \iff U_2(A) < U_2(B).$$

That is, in this case, a decision maker will not choose A in both cases.

For (b), note that

$$U_1(A) > U_1(B) \iff 0.51 > \min\{\pi : \pi \in P\};$$

$$U_2(A) > U_2(B) \iff 0.49 > \min\{\pi : \pi \in P\} \iff 0.51 > \max\{\pi : \pi \in P\}.$$

That is, for every set $P \subset [0, 1]$ such that $\min P < 0.51 < \max P$, we have

$$U_1(A) > U_1(B) \text{ and } U_1(A) > U_2(B).$$

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[2] i.e. when there is a single belief, a decision maker with a von Neumann-Morgenstern utility function will not choose box 1 in both cases.