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A LIMIT THEOREM ON THE CORE OF AN ECONOMY*

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1. INTRODUCTION

IN his *Mathematical Psychics* [5], Edgeworth presented a remarkable study of the exchanges of two commodities that might arise in an economy with two types of consumers. The first case that he considers concerns two individuals each of whom initially possesses certain quantities of each commodity. The result of trading consists of a reallocation of the total amounts of the two commodities and may, therefore, be described geometrically by a point in the Edgeworth box corresponding to that economy.

Edgeworth confines his attention to those exchanges which are Pareto optimal, i.e., those which cannot yield greater satisfaction for one consumer without impairing that of the other by means of additional trade. He further restricts the admissible final allocations to those which are at least as desired by *both* consumers as the allocation prevailing before trading. Those allocations which are not ruled out by either of these considerations constitute the "contract curve."

As Edgeworth remarks, a competitive allocation is on the contract curve (under assumptions listed in Section 2). But so are many other allocations, and nothing in the analysis of the case of two consumers indicates that the competitive solutions play a privileged role. In order to single out the competitive allocations Edgeworth introduces an expanded economy which consists of $2n$ consumers divided into two types; everyone of the same type having identical preferences and identical resources before trading takes place. The object is to demonstrate that as n becomes large, more and more allocations are

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ruled out, and eventually only the competitive allocations remain. This statement can be paraphrased by saying that the contract curve shrinks to the set of competitive equilibria as the number of consumers becomes infinite.

It is clear that the two principles mentioned above for ruling out allocations must be supplemented by some additional principles if this result is to be correct. The general principle which Edgeworth formulated is that of "recontracting." Consider an allocation of the total resources of the $2n$ consumers and consider any collection of consumers (which need not include the same number of each type). This collection "recontracts out," if it is possible for its members to redistribute their initial resources among themselves in such a way that some member of the collection prefers the new outcome to the allocation previously given while no member desires it less. The presumption is that an allocation is not made if it can be recontracted out by some group of consumers.

Edgeworth shows that the set of allocations which are not recontracted out decreases as n increases, and has the set of competitive equilibria as a limit. The proof given in *Mathematical Psychics* could easily be rewritten in the style of contemporary mathematical economics. It is, however, based on the geometrical picture of the Edgeworth box and does not seem to be applicable to the general case involving more than two commodities and more than two types of consumers.

As Martin Shubik pointed out, the question can be studied from the point of view of n-person game theory. In a very stimulating paper **[12]** he analyzed the Edgeworth problem, using the von Neumann-Morgenstern concept of a solution, and also Gillies' [6] concept of the "core." Other discussions of markets as n-person games may be found in von Neumann and Morgenstern [7] and in several papers by Shapley **[9, 10]**.

In all these contributions, extensive use is made of a transferable utility. While this concept has been readily accepted in game theory, it has remained foreign to the mainstream of economic thought. Some recent work has been done, however, on a version of n-person game theory which avoids the assumption of transferable utility [1, 2] and which includes a definition of the core. It is this concept which corresponds to the Edgeworth notion of recontracting.

In [8] Scarf analyzed the core in the latter sense in an economy with an arbitrary number of types of consumers and an arbitrary number of commodities. Economies consisting of r consumers of each type were considered and it was proved that an allocation which assigns the same commodity bundle to all consumers of the same type and which is in the core for all r must be competitive. An economy consisting of an infinite sequence of consumers of each type was also studied and it was demonstrated that an allocation in the core of this economy is competitive. A suggestion for a simplification of the proofs of these theorems and for a weakening of their assumptions was given by Debreu [4].

Our main purpose is to show that the first of the two theorems mentioned in the last paragraph is very widely applicable and, thereby, to obtain a further considerable simplification of the study of the core and to discard an awkward assumption used in both papers [8, (Section 4, A.2)] and [4, (A.4)]. Our second purpose is to cover a case in which production is possible.

In the traditional Walrasian analysis of equilibrium the resources of the consumers and their shares in the producers' profits are specified. All the agents of the economy are assumed to adapt themselves to a price system which one then tries to choose so as to equate total demand and total supply. In the Paretian study of optimality, prices are seen from a second and very different point of view. The problem of efficient organization of an economy with an unspecified distribution of resources is considered, and it is essentially shown that a state of the economy is an optimum one if and only if there exists a price system to which every consumer and every producer is adapted. In Edgeworth's theorem, and in the generalization that we present here, prices appear in a third and again very different light. Given an economy with a specified distribution of resources composed of a certain number of types of consumers which is small relative to the numbers of consumers of each type, an outcome is viable, i.e. no coalition can block it, if and only if there exists a price system to which consumers and producers are adapted. That is to say, competitive equilibria, and only they, are viable. As in the study of Pareto optima, prices emerge from the analysis in a situation in which they were not introduced *a priori*.

2. THE CORE IN A PURE EXCHANGE ECONOMY

At first we study an economy in which no production can take place. We consider m consumers each with specific preferences for commodity bundles consisting of nonnegative quantities of a finite number of commodities. Such a commodity bundle is represented by a vector in the nonnegative orthant of the commodity space, and the preferences of the i-th consumer by a complete preordering, $\precsim_i$. The interpretation of $x' \gtrsim_i x$ is, of course, that the i-th consumer either prefers x' to x or is indifferent between them. If x' is strictly preferred to x, then we write $x' \succ_i x$.

Three assumptions will be made on the preferences:

1. *Insatiability*. Let x be an arbitrary nonnegative commodity bundle. We assume that there is a commodity bundle x' such that $x' \succ_i x$.

2. *Strong-convexity*. Let x' and x be arbitrary different commodity bundles with $x' \gtrsim_i x$, and let α be an arbitrary number such that $0 < \alpha < 1$. We assume that $\alpha x' + (1 - \alpha)x \succ_i x$.

3. *Continuity*. We assume that for any nonnegative x', the two sets

$$\{x \mid x \underset{i}{\gtrsim} x'\} \quad \text{and} \quad \{x \mid x \underset{i}{\lesssim} x'\}$$

are closed.

Each consumer owns a commodity bundle which he is interested in exchanging for preferred commodity bundles. The vector ω_i will represent the resources of the i-th consumer. We find it convenient to make the following assumption:

4. *Strict positivity of the individual resources*. We assume that every consumer owns a strictly positive quantity of every commodity.

The core can now be defined. Since production is not considered in the present section, the result of trading consists of an allocation of the total supply $\sum_{i=1}^{m} \omega_i$, and is therefore described by a collection of m nonnegative commodity bundles $(x_1, \cdots, x_m)$ such that

$$\sum_{i=1}^{m} (x_i - \omega_i) = 0 \ .$$

An allocation is in the core if it cannot be recontracted out by any set of consumers S, i.e. if no set of consumers S can redistribute their own initial supply among themselves so as to improve the position of any one member of S without deterioration of that of any other. We emphasize here that it is permissible for an arbitrary set of consumers to combine and reallocate their own assets independently of the remaining consumers in the economy.

To give a formal definition of the core we introduce the notion of set of consumers blocking an allocation. Let $(x_1, \cdots, x_m)$ with $\sum_{i=1}^{m} (x_i - \omega_i) = 0$ be an assignment of the total supply to the various consumers, and let S be an arbitrary set of consumers. We say that the allocation is blocked by S if it is possible to find commodity bundles x'_i for all i in S such that

$$(1) \qquad \sum_{i \in S} (x'_i - \omega_i) = 0 \, ,$$

and

$$(2) \qquad x'_i \underset{i}{\gtrsim} x_i \, ,$$

for all i in S, with strict preference for at least one member of S.

The core of the economy is defined as the collection of all allocations of the total supply which cannot be blocked by any set S. One immediate consequence of this definition is that an allocation in the core is Pareto optimal. We prove this by taking for the set S all consumers. On the other hand, if we take the possible blocking set to consist of the i-th consumer himself, then we see that an allocation in the core must satisfy the condition $x_i \gtrsim_i \omega_i$; i.e., the i-th consumer does not prefer his initial holding to the commodity bundle that he receives on the basis of an allocation in the core. Many other conditions will, of course, be obtained as more general sets S are considered.

It is not clear that there always will be some allocations in the core. One can easily construct examples in n-person game theory in which every imputation is blocked by some coalition so that the core is empty. Economies with an empty core may also be found if the usual assumptions on preferences are relaxed. The following example due to Scarf, Shapley, and Shubik is typical.

Consider an economy with two commodities and three consumers,

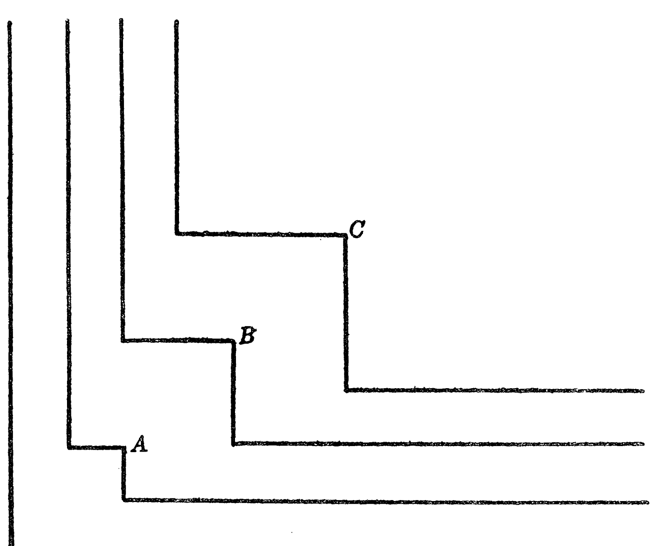

FIGURE 1

each of whom has preferences described by the indifference curves in Figure 1. It is shown in [11] that if the initial resources of each consumer consist of one unit of each commodity, then the core of the resulting economy is empty. This conclusion does not depend on the lack of smoothness of the indifference curves.

In this paper we make the customary assumptions listed above, in which case it may be shown that the core is not empty. The procedure for doing this is to observe that a competitive allocation exists, and then to demonstrate that every competitive allocation is in the core.

It is known that, given our four assumptions on preferences and initial holdings, there is a competitive equilibrium [3]. That is to say, there are nonnegative commodity bundles $x_1, \cdots, x_m$ with $\sum_{i=1}^{m} (x_i - \omega_i) = 0$ and a price vector p, such that x_i satisfies the preferences of the i-th consumer subject to the budget constraint $p \cdot x_i \leqq p \cdot \omega_i$. The familiar argument of welfare economics by which a competitive allocation is proved to be Pareto optimal has been extended, as follows, by Shapley to prove:

THEOREM 1. *A competitive allocation is in the core.*

First notice that $x'_i \succ_i x_i$ obviously implies $p \cdot x'_i > p \cdot \omega_i$. For, otherwise, x_i does not satisfy the preferences of the i-th consumer under his budget constraint. Notice also that $x'_i \succsim_i x_i$ implies $p \cdot x'_i \geqq p \cdot \omega_i$. For, if $p \cdot x'_i < p \cdot \omega_i$, there is, according to our assumptions 1 and 2, a consumption vector in a neighborhood of x'_i that satisfies the budget constraint and that is preferred to x_i.

Let S be a possible blocking set, so that $\sum_{i \in S} (x'_i - \omega_i) = 0$ with $x'_i \succsim_i x_i$ for all i in S, and with strict preference for at least one i. From the two remarks we have just made, $p \cdot x'_i \geqq p \cdot \omega_i$ for all i in S, with strict inequality for at least one i. Therefore

$$\sum_{i \in S} p \cdot x'_i > \sum_{i \in S} p \cdot \omega_i \,,$$

a contradiction of $\sum_{i \in S} (x'_i - \omega_i) = 0$.

3. THE CORE AS THE NUMBER OF CONSUMERS BECOMES INFINITE

We shall now follow the procedure first used by Edgeworth for enlarging the market. We imagine the economy to be composed of m types of consumers, with r consumers of each type. For two consumers to be of the same type, we require them to have precisely the same preferences and precisely the same vector of initial resources. The economy therefore consists of mr consumers, whom we index by the pair of numbers (i, q), with $i = 1, 2, \cdots, m$ and $q = 1, 2, \cdots, r$.

The first index refers to the type of the individual and the second index distinguishes different individuals of the same type.

An allocation is described by a collection of mr nonnegative commodity bundles x_{iq} such that

$$\sum_{i=1}^{m}\sum_{q=1}^{r} x_{iq} - r\sum_{i=1}^{m} \omega_i = 0 .$$

The following theorem makes for the simplicity of our study:

THEOREM 2. *An allocation in the core assigns the same consumption to all consumers of the same type.*

For any particular type i, let x_i represent the least desired of the consumption vectors x_{iq} according to the common preferences for consumers of this type and assume that for some type i' two consumers have been assigned different commodity bundles. Then

$$\frac{1}{r}\sum_{q=1}^{r} x_{iq} \underset{i}{\gtrsim} x_i , \qquad \text{for all } i ,$$

with strict preference holding for i'. However,

$$\sum_{i=1}^{m}\left(\frac{1}{r}\sum_{q=1}^{r} x_{iq} - \omega_i\right) = 0 ,$$

and therefore the set consisting of one consumer of each type, each of whom receives a least preferred consumption, would block.

The theorem we have just proved implies that an allocation in the core for the repeated economies considered here may be described by a collection of m nonnegative commodity bundles $(x_1, \cdots, x_m)$ with $\sum_{i=1}^{m}(x_i - \omega_i) = 0$. The particular collections of commodity bundles in the core will, of course, depend on r. It is easy to see that the core for $r + 1$ is contained in the core for r, for a coalition which blocks in the economy with r repetitions will certainly be available for blocking in the economy with $(r + 1)$ repetitions.

If we consider a competitive allocation in the economy consisting of one participant of each type and repeat the allocation when we enlarge the economy to r participants of each type, the resulting allocation is competitive for the larger economy and consequently is in the core. We see, therefore, that as a function of r, the cores form a nonincreasing sequence of sets, each of which contains the collection of competitive allocations for the economy consisting of one consumer of each type. Our main result is that no other allocation is in the core for all r.

THEOREM 3. *If $(x_1, \cdots, x_m)$ is in the core for all r, then it is a* *competitive allocation.*

Let Γ_i be the set of all z in the commodity space such that $z + \omega_i \succ_i x_i$, and let Γ be the convex hull of the union of the sets Γ_i. Since, for every i, Γ_i is convex (and nonempty), Γ consists of the set of all vectors z which may be written as $\sum_{i=1}^m \alpha_i z_i$, with $\alpha_i \geqq 0$, $\sum_i^m \alpha_i = 1$, and $z_i + \omega_i \succ_i x_i$. The following diagram describes this set in the case of two commodities and two types of consumers.

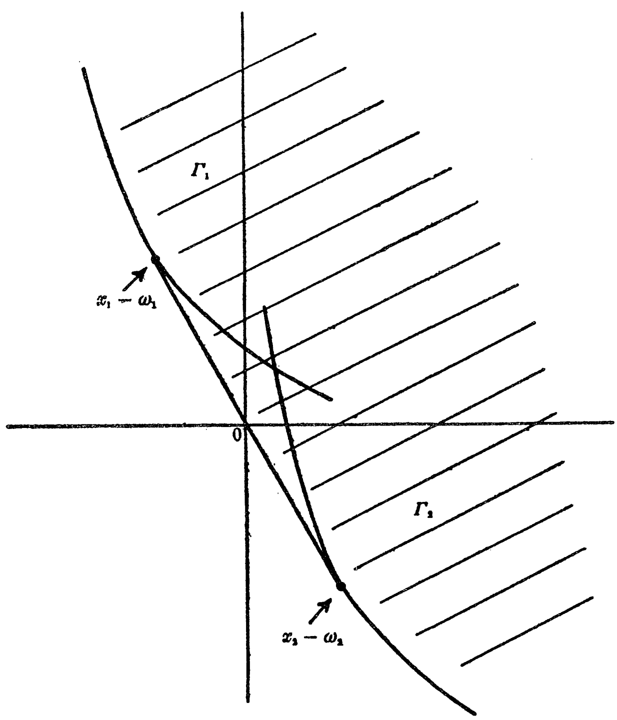

FIGURE 2

We first form the set of commodity bundles which are preferred to x_1 and from each of these subtract the vector ω_1, obtaining the set Γ_1. We do the same with x_2 and ω_2 in order to obtain Γ_2. We then take the union of Γ_1 and Γ_2 and form the convex hull obtaining Γ. Verifying that the origin does not belong to the set Γ in the general case of an arbitrary number of commodities and an arbitrary number of types of consumers is the key step in the proof of Theorem 3.

Let us suppose that the origin belongs to Γ. Then $\sum_{i=1}^m \alpha_i z_i = 0$ with $\alpha_i \geqq 0$, $\sum_{i=1}^m \alpha_i = 1$ and $z_i + \omega_i \succ_i x_i$. Select an integer k, which will eventually tend to $+\infty$, and let a_i^k be the smallest integer greater than or equal to $k\alpha_i$. Also let I be the set of i for which $\alpha_i > 0$.

For each i in I we define z_i^k to be $[(k\alpha_i)/(a_i^k)]z_i$ and observe that $z_i^k + \omega_i$ belongs to the segment $[\omega_i, z_i + \omega_i]$ and tends to $z_i + \omega_i$ as k tends to infinity. The continuity assumption on preferences implies that $z_i^k + \omega_i \succ_i x_i$ for sufficiently large k. Moreover,

$$\sum_{i \in I} a_i^k z_i^k = k \sum_{i \in I} \alpha_i z_i = 0 .$$

Consider the coalition composed of a_i^k members of type i to each one of whom we assign $\omega_i + z_i^k$, where i runs over the set I. Such a coalition blocks the allocation $(x_1, \cdots, x_m)$ repeated a number of times equal to $\max_{i \in I} a_i^k$. This contradicts the assumption that $(x_1, \cdots, x_m)$ is in the core for all r.

We have, therefore, established that the origin does not belong to the convex set Γ. Consequently, there is a hyperplane through it with normal p such that $p \cdot z \geqq 0$ for all points z in Γ.

If $x' \succ_i x_i$, then $x' - \omega_i$ is in Γ_i, hence in Γ, and we obtain $p \cdot x' \geqq p \cdot \omega_i$. Since in every neighborhood of x_i there are consumptions strictly preferred to x_i, we also obtain $p \cdot x_i \geqq p \cdot \omega_i$. But

$$\sum_{i=1}^{m} (x_i - \omega_i) = 0 .$$

Therefore $p \cdot x_i = p \cdot \omega_i$ for every i.

The argument is virtually complete at this stage. We have demonstrated the existence of prices p such that for every i, (1) $x' \succ_i x_i$ implies $p \cdot x' \geqq p \cdot \omega_i$ and (2) $p \cdot x_i = p \cdot \omega_i$. As is customary in equilibrium analysis, there remains to show that x_i actually satisfies the preferences of the i-th consumer subject to his budget constraint, i.e. that $x' \succ_i x_i$ actually implies $p \cdot x' > p \cdot \omega_i$. Since ω_i has all of its components strictly positive, there is a nonnegative x^0 strictly below the budget hyperplane. If for some x'', both $x'' \succ_i x_i$ and $p \cdot x'' = p \cdot \omega_i$, the points of the segment $[x^0, x'']$ close enough to x'' would be strictly preferred to x_i and strictly below the budget hyperplane, a contradiction of (1). This completes the demonstration of Theorem 3.

4. THE CORE IN A PRODUCTIVE ECONOMY

An entirely straightforward extension of our results on the core to an economy in which production is possible can be given. We assume that all coalitions of consumers have access to the same production possibilities described by a subset Y of the commodity space. A point y in Y represents a production plan which can be carried out. Inputs into production appear as negative components of y and outputs as positive components. From now on, in addition to the four conditions given in Section 2 (insatiability, strong-convexity and continuity of preferences, and strict positivity of the individual resources), we impose on the economy the following condition:

5. *Y is a convex cone with vertex at the origin.*

Thus Sections 2 and 3 dealt with the particular case where the cone Y is degenerate to the set having the origin as its only element.

In the new context, an allocation for an economy with m consumers is a collection of nonnegative commodity bundles $(x_1, \cdots, x_m)$ such that there is in Y a production plan y satisfying the equality of demand and supply $\sum_{i=1}^m x_i = y + \sum_{i=1}^m \omega_i$, i.e., such that $\sum_{i=1}^m (x_i - \omega_i)$ belongs to Y. This allocation is blocked by the set S of consumers if it is possible to find commodity bundles x_i' for all i in S such that (1) $\sum_{i \in S} (x_i' - \omega_i)$ belongs to Y, and (2) $x_i' \succsim_i x_i$ for all i in S, with strict preference for at least one member of S. The core of the economy is defined as the collection of all allocations which cannot be blocked.

An allocation is competitive if there exists a price system p such that the profit is maximized on Y (since Y is a cone with vertex at the origin, the maximum profit is zero) and that x_i satisfies the preferences of the i-th consumer under the constraint $p \cdot x \leqq p \cdot \omega_i$. Assumptions 1—5 are no longer sufficient to insure the existence of a competitive allocation, but Theorem 1 remains true: A competitive allocation is in the core.

The proof hardly differs from the one we have given. The two opening remarks are unchanged. Let S be a possible blocking set, so that $\sum_{i \in S} (x_i' - \omega_i) = y$ in Y with $x_i' \succsim_i x_i$ for all i in S, with strict preference for at least one i, and with $p \cdot y \leqq 0$. Since $p \cdot x_i' \geqq p \cdot \omega_i$ for all i in S, with strict inequality for at least one i, we have $\sum_{i \in S} p \cdot x_i' > \sum_{i \in S} p \cdot \omega_i$, or $p \cdot y > 0$, a contradiction.

As before we consider an economy composed of m types of consumers with r consumers of each type. An allocation is described by a collection of mr commodity bundles x_{iq} such that $\sum_{i=1}^m \sum_{r=1}^q x_{iq} - r \sum_{i=1}^m \omega_i$ belongs to Y. It is a simple matter to verify the analogue of Theorem 2: *An allocation in the core assigns the same consumption to all consumers of the same type.* The only modification in the previous proof involves the fact that $y \in Y$ implies $(1/r)y \in Y$.

The allocations in the cores may therefore be described by a collection of m commodity bundles $(x_1, \cdots, x_m)$ with $\sum_{i=1}^m (x_i - \omega_i)$ in Y. Again it is clear that the cores form a nonincreasing sequence of sets as r increases. We now indicate the proof of the analogue of

Theorem 3: *If* $(x_1, \cdots, x_m)$ *is in the core for all* r, *then it is a competitive allocation.* The set Γ is defined, as before, to be the convex hull of the union of the m sets

$$\Gamma_i = \left\{z \mid z + \omega_i \succ_i x_i\right\} .$$

We then show that Γ and Y are disjoint. Suppose, to the contrary, that $\sum_{i=1}^m \alpha_i z_i = y$ in Y with $\alpha_i \geqq 0, \sum_{i=1}^m \alpha_i = 1$, and $z_i + \omega_i \succ_i x_i$. Using the same definitions of k, a_i^k, I and z_i^k as in the proof of Theorem 3, we see that $z_i^k + \omega_i \succ_i x_i$ for sufficiently large k. Moreover,

$$\sum_{i \in I} a_i^k z_i^k = k \sum_{i \in I} \alpha_i z_i = ky .$$

Since $ky \in Y$, the allocation is blocked by the coalition we have described in proving Theorem 3. Thus a contradiction has been obtained.

The two convex sets Γ and Y may, therefore, be separated by a hyperplane with normal p such that $p \cdot z \geqq 0$ for all points z in Γ and $p \cdot y \leqq 0$ for all points y in Y. The demonstration then proceeds as before to verify that we indeed have a competitive allocation.

5. GENERALIZATIONS

Until now we have constrained the consumption bundles of consumers to belong to the nonnegative orthant of the commodity space. This restriction, which was made only to keep the exposition as simple as possible, is not essential. Instead we can require the consumptions of all the consumers of the i-th type $(i = 1, \cdots, m)$ to belong to a given subset X_i of the commodity space. We impose on these consumption sets the condition:

0. X_i *is convex.*

We make the appropriate modifications on Assumptions 1—4; in particular, in 3, the two sets $\{x \mid x \succsim_i x'\}$ and $\{x \mid x \precsim_i x'\}$ are now assumed to be closed in X_i, and in 4, ω_i is now assumed to be interior to X_i. Then the three theorems are established without alteration of their proofs.

A second generalization consists in replacing Assumption 2 (strong-convexity of preferences) by

2′. *Convexity.* Let x' and x be arbitrary commodity bundles with $x' \succ_i x$, and let α be an arbitrary number such that $0 < \alpha < 1$. We assume that $\alpha x' + (1 - \alpha)x \succ_i x$.

This substitution affects neither the statement nor the proof of Theorem 1. In order to establish the analogue of Theorem 2, we consider an economy with r consumers of each one of m types. Given an allocation (x_{iq}) in its core, we define $\bar{x}_i$ to be $(1/r) \sum_{q=1}^r x_{iq}$ and, as before, we denote by x_i the least desired of the consumption bundles x_{iq} according to the common preferences for consumers of the i-th type. Since $\sum_{i=1}^{m} (\bar{x}_i - \omega_i)$ belongs to Y, the coalition consisting of one consumer of each type who receives a least preferred consumption blocks, unless $\bar{x}_i \underset{i}{\sim} x_i$ for every i. Therefore, by 2′, *an allocation in the core assigns to all consumers of the same type consumptions indifferent to the average of the consumptions for that type.* This suggests defining the *strict core* of the economy as the collection of all unblocked allocations with the same consumption bundles assigned to all consumers of the same type. As we have just seen, with any allocation in the core is associated an allocation (consisting of the m average consumptions repeated r times) in the strict core which is indifferent to the first allocation for every consumer. Thus the distinction between the core and the strict core is not essential. However, we can treat the strict core under 2′ exactly as we treated the core under 2. As a function of r, the strict cores form a non-increasing sequence of sets and *if* $(x_1, \cdots, x_m)$ *is in the strict core for all* r, *then it is a competitive allocation.*

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