Lecture 1: Sets, Relations and Orders

Advanced Microeconomics I, ITAM

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1 Symbols

First, we list a number of symbols we will repeatedly use in this course.

- $\forall$: for all
- $\exists$: exists
- $\nexists$: does not exist
- $\exists!$: exists uniquely
- $:=$: by definition
- e.g.: for example
- i.e.: that is
- c.f.: compared to
- s.t.: such that

Probably the most important symbol is $\Longrightarrow$, which means imply. By $P \Longrightarrow Q$, we mean whenever P is satisfied, Q is also satisfied.

$\Longleftrightarrow$ is usually referred to with different names. It could mean if and only if (iff), equivalent to, or necessary and sufficient condition. In particular, $P \Longleftrightarrow Q$ means both $P \Longrightarrow Q$ and $Q \Longrightarrow P$ hold.

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2 Sets

A *set* is a collection of elements. Usually, there are two ways to write a set:

- By enumeration: $S = \{x_1, ..., x_n\}$ or $S = \{x_1, ..., x_n, ...\}$
- By description: $S = \{x : x \text{ satisfies some properties}\}$

Whenever an element x is in a set S, we write $x \in S$. Otherwise, we write $x \notin S$.

Next, we give some examples of sets.

Example 1. *Here, we list some sets:*

- $\emptyset$*: empty set*

  e.g. $\{x : x \neq x\}$.

- $\mathbb{R}$*: real numbers, or numbers*
- $\mathbb{Z}$*: integers* $\{0, \pm 1, \pm 2, ...\}$
- $\mathbb{N}$*: natural numbers* $\{1, 2, ...\}$
- $\mathbb{Q}$*: rational numbers* $\{\frac{a}{b} : a, b, \in \mathbb{Z}, b \neq 0\}$
- *Unit Balls in Euclidean spaces* $\mathbb{R}^n$

  e.g. in $\mathbb{R}^n$, $\{x \in \mathbb{R}^n : |x| \leq 1\}$

  in $\mathbb{R}^n$, $\{x \in \mathbb{R}^n : |x| < 1\}$

2.1 Set Operations

Given two sets A and B, we say A is a *subset* of B, denoted by $A \subset B$, if for all $x \in A$, we have $x \in B$.

The *power set* of a set A, denoted by 2^A or $\mathcal{P}(A)$, is the set consisting of all subsets of A. e.g. When $A = \{1, 2\}$, $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. It is a good conceptual exercise to know the power set of the empty set.

Exercise. *What is the power set of* $\emptyset$*? Is it empty?*

We say set A is *equal* to set B if both $A \subset B$ and $B \subset A$ hold.

Exercise. *Prove that* $\{1\} = \{1, 1\}$, *and* $\{1, 2\} = \{2, 1\}$.

The *intersection* of set A and set B is defined by

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

The *union* of set A and set B is defined by

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

The *complement* of a set A is defined by

$$A^c = \{x : x \notin A\}.$$

Knowing the intersection (union) of two sets, we are able to define the intersection (union) of finitely many sets. For instance, for three sets A, B, C, their intersection $A \cap B \cap C$ is defined by

$$A \cap B \cap C = (A \cap B) \cap C$$

We note here $A \cap B$ is a set, and the right hand side is the intersection of two sets, set $A \cap B$ and set C. Intuitively, the intersection of three sets should not depend on the order we write them. That is, the intersection of A, B, C should have no difference with the intersection of B, C, A. Therefore, after writing such a definition, we should check it is well defined:

Exercise. *Prove that for sets* A, B, C, *we have*

$$(A \cap B) \cap C = A \cap (B \cap C)$$

For intersection (union) of more sets, we could use the above argument to define inductively. However, this inductive argument only works when we study the intersection (union) of finitely many sets. For the intersection (union) of infinitely many sets, its definition is as follows:

Given an index set $\mathcal{A}$, and for each index $\alpha \in \mathcal{A}$, S_α is a set. We define the intersection and union of the class $\{S_\alpha : \alpha \in \mathcal{A}\}$ respectively by

$$\bigcap_{\alpha \in \mathcal{A}} S_\alpha = \{x : x \in S_\alpha \text{ for all } \alpha \in \mathcal{A}\},$$

$$\bigcup_{\alpha \in \mathcal{A}} S_\alpha = \{x : x \in S_\alpha \text{ for some } \alpha \in \mathcal{A}\}.$$

To better understand the intersection and union of infinitely many sets, we give two examples. Recall the definition of intervals on $\mathbb{R}$:

$$(a, b) = \{x \in \mathbb{R} : x > a \text{ and } x < b\}$$

$$[a, b) = \{x \in \mathbb{R} : x \geq a \text{ and } x < b\}$$

$$(a, b] = \{x \in \mathbb{R} : x > a \text{ and } x \leq b\}$$

$$[a, b] = \{x \in \mathbb{R} : x \geq a \text{ and } x \leq b\}$$

Example 2. *When* $\mathcal{A} = (0, 1)$*, and for all* $\alpha \in (0, 1)$*,* $S_\alpha = (-\alpha, \alpha)$*, we have*

$$\bigcap_{\alpha \in \mathcal{A}} S_\alpha = \{0\}$$

.

Example 3. *When* $\mathcal{A} = (0, 1)$*, and for all* $\alpha \in (0, 1)$*,* $S_\alpha = [-\alpha, \alpha]$*, we have*

$$\bigcup_{\alpha \in \mathcal{A}} S_\alpha = (-1, 1)$$

.

Exercise 1. *Verify the above two examples are correct.* [1]

For the composition of intersections, unions, and complement, see the following exercise.

Exercise 2. *Prove the following equations.* [2]

- $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
- $(A^c)^c = A$
- $(\bigcap_{\alpha \in \mathcal{A}} A_\alpha)^c = \bigcup_{\alpha \in \mathcal{A}} A_\alpha^c$
- $(\bigcup_{\alpha \in \mathcal{A}} A_\alpha)^c = \bigcap_{\alpha \in \mathcal{A}} A_\alpha^c$

[1] Hint: for the first, we need to show 0 is in the intersection, and no other elements is in the intersection.
[2] Hint: draw Venn diagrams.

2.2 Cardinality

Sometimes, we are interested to know the number of elements a set contains. Such a number is referred to as the cardinality of this set. Formally, the *cardinality* of set S, denoted by $|S|$, is defined as follows:

- when $S = \emptyset$, $|S| = 0$;
- when S can be written by enumerate in the form $S = \{x_1, ..., x_n\}$, where x_i and x_j are mutually disjoint, we say $|S| = n$;
- Otherwise, $|S| = +\infty$.

In particular, if the cardinality of set S is 1, i.e. S contains only 1 element, we say set S is a *singleton*. When the cardinality of set S is not $+\infty$, we say set S is a *finite set*.

Exercise. *For any finite set S, prove that*

$$|\mathcal{P}(S)| = 2^{|S|}$$

Remark. *The above exercise gives a reason on why sometimes we write a power set as 2^S.*

2.3 Partition

A *partition* of set X is a set of pairwisely disjoint, nonempty subset of X, such that their union is X.

For instance, when $X = \{1, 2, 3\}$, $\{\{1, 2\}, \{3\}\}$ is a partition of X, but $\{\{1, 2\}, \{2, 3\}\}$ or $\{\{1\}, \{2\}\}$ is not a partition of X. Sometimes, it helps to think as follows: there are three balls, with names 1,2,3 on a table. A partition of this set corresponds to that we separate these balls into different piles. In the first case, the balls are separate to two piles, the first with balls 1 and 2, the second with ball 3. In the second case, it is not a partition as ball 2 is used two times. In the third case, it is not a partition as ball 3 is not used.

3 Relation

From time to time, we compare things: in mathematics, the numbers are compared with each other; in economics, people compare different boxes of fruits (and then decide what to eat). In this section, we define the object of comparison.

3.1 Cartesian Product

First, as we have seen, the set $\{1, 2\}$ is equal to the set $\{2, 1\}$. That is, the order of elements in a set is not taken into account under the current definition. Therefore, we need to define ordered pair as follows:

An *ordered pair* (a, b) is an ordered list consisting a and b such that [3]

$$(a, b) = (c, d) \text{ if and only if } a = b \text{ and } c = d$$

Inductively[4], we can define any *sequence* of finite length $(a_1, a_2, ..., a_n)$. Note,

$$(a_1, ..., a_n) = (b_1, ..., b_n) \text{ if and only if } a_k = b_k \text{ for all } 1 \leq k \leq n$$

3.2 Relation

Given non-empty sets X, Y, a subset $R \subset X \times Y$ is called a *relation from X to Y*. When $Y = X$, we say R is a *relation on X*.

Example 4. *On $\mathbb{R}$, $>$ is a relation defined by the set*

$$>:= \{(x, y) \in \mathbb{R}^2 : x > y\}^{5}$$

In particular, if $(x, y) \in >$, we write $x > y$. For instance, $(3, 2) \in >$, so we write $3 > 2$.

[3] More rigorously, (a, b) is defined by the set $\{\{a\}, \{a, b\}\}$. However, such an understanding is not required in this course. For interested readers, it is a good exercise to prove under such set theoretic definition, we have $(a, b) = (c, d)$ if and only if $a = b$ and $c = d$.

[4] For instance, (a, b, c) can be defined as $((a, b), c)$.

[5] This definition seems to be not well-defined as it seems we are defining the notation $>$ by using it. A better description of the set on the right hand set would be $\{(x, x - a) : x \in \mathbb{R}, a \text{is } \text{ a positive number}\}$, where the set of positive numbers can be defined by the limit of rational numbers p/q where $p, q \in \mathbb{N}$.

Next, we list a number of desired properties on a relation:

1. R is *reflexive* if for all $x \in X$, $(x, x) \in R$.

   e.g. $=, \geq$ on $\mathbb{R}$ are reflexive; $>$ on $\mathbb{R}$ is not reflexive.

2. R is *symmetric* if for all $x, y \in X$, whenever $(x, y) \in R$, we have $(y, x) \in R$.

   e.g. $=, \neq$ on $\mathbb{R}$ are symmetric; $>, \geq$ on $\mathbb{R}$ are not symmetric.

3. R is *transitive* if for all $x, y, z \in X$, whenever $(x, y) \in R, (y, z) \in R$, we have $(x, z) \in R$.

   e.g. $=, \geq, >$ on $\mathbb{R}$ are transitive; $\neq$ on $\mathbb{R}$ is not transitive.

4. R is *complete* if for all $x, y \in X$, either $(x, y) \in R$ or $(y, x) \in R$.

   e.g. $\geq$ on $\mathbb{R}$ is complete; $=, \neq$ on $\mathbb{R}$ are not complete.

   Notice that a complete relation must be reflexive. (Why?)

In the following subsections, we study the two important refinements of relations.

3.3 Equivalence Relation

We say a relation $\sim$ on X is an *equivalence relation* if it is reflexive, symmetric and transitive.

Exercise. *Find examples of the following relations,*

1. *A relation that is reflexive, symmetric, but is not transitive.*

2. *A relation that is reflexive, transitive, but is not symmetric.*

3. *A relation that is symmetric, transitive, but is not .*

Exercise 3. $=$ *on* $\mathbb{R}$ *is an equivalence relation.*

Example 5. *On* $\mathbb{R}^2_{++} = \{(x, y) \in \mathbb{R} : x, y > 0\}$, *we define a relation* $\sim$ *such that*

$$(x_1, y_1) \sim (x_2, y_2) \text{ if and only if } \sqrt{x_1 y_1} = \sqrt{x_2 y_2}$$

It is an exercise to check $\sim$ *is an equivalence relation.*

An equivalence relation gives us a new way to view the set X. To describe this new way, we define the equivalence class:

For any $x \in X$, the *equivalence class of x relative to $\sim$ is*

$$[x]_\sim = \{y \in X : y \sim x\}$$

Example. *(continued..) Given $\sim$ defined in the previous example, we note $[(1,1)]_\sim = \{(x,y) \in \mathbb{R}^2_{++} : xy = 1\}$ That is, the equivalence class of point $(1,1)$ corresponds to a hyperbola in the first quadrant of the plane. (Please complete the graph by yourself.)*

Exercise. *Given an equivalence relation $\sim$, the set of equivalence classes*

$$\{[x]_\sim : x \in X\}$$

is a partition of X.

Remark. *The set $\{[x]_\sim : x \in X\}$ is called a quotient space, denoted by $X/\sim$.*

Exercise. *On $\mathbb{R}^2_{++}$, we define a relation $\sim$ such that*

$$(x_1, y_1) \sim (x_2, y_2) \text{ if and only if } f(x_1, y_1) = f(x_2, y_2)$$

For the following cases, determine whether $\sim$ is an equivalence relation. If it is, draw an equivalence relation on $\mathbb{R}^2$.

1. $f(x,y) = \log x + 2\log y$
2. $f(x,y) = \max\{x,y\}$
3. $f(x,y) = \min\{x,y\}$
4. $f(x,y) = x + \log y$

3.4 Preorders and Orders

We note that an equivalence relation helps to describe the case that an agent is indifferent between two bundles. However, at times, an agent also needs to compare two bundles. Therefore, we need to define the order relation:

We say a relation R on X is a *preorder* if it is reflexive and transitive. In addition, we say a relation R on X is an *order* if it is a preorder and satisfies

$$\text{Whenever } (x,y) \in R, (y,x) \in R, \text{ we have } x = y$$

Example 6. *The equivalence relation* $\sim$ *on* $\mathbb{R}^2_{++}$ *in the previous subsection is a preorder, but is not an order.*

To see it, we first notice that any equivalence relation must be a preorder by definition. Next, we note that while $(2,2) \sim (1,4)$ *and* $(1,4) \sim (2,2)$*, we do not have* $(2,2) = (1,4)$ *in* $\mathbb{R}^2_{++}$.

Next, we give an example of order relation.

Example 7. *(*$\geq$ *on* $\mathbb{R}^n$*) The relation* $\geq$ *on* $\mathbb{R}^n$ *is defined as follows:*

- *We write* $(x_1, ..., x_n) \geq (y_1, ..., y_n)$ *if* $x_1 \geq y_1$*,...,* $x_n \geq y_n$.

To see $\geq$ *is an order relation, we check the two conditions in the definition. First, to see it is a preorder, we notice: firstly, for all* $(x_1, ..., x_n) \in \mathbb{R}^n$, $(x_1, ..., x_n) \geq (x_1, ..., x_n)$*. i.e.* $\geq$ *is reflexive. Moreover, if* $(x_1, ..., x_n) \geq (y_1, ..., y_n)$ *and* $(y_1, ..., y_n) \geq (z_1, ..., z_n)$*, we have* $(x_1, ..., x_n) \geq (z_1, ..., z_n)$*. i.e.* $\geq$ *is transitive. Next, we prove it satisfies the condition*

$$\textit{Whenever } (x,y) \in R, (y,x) \in R, \textit{ we have } x = y$$

That is, take any vectors $(x_1, ..., x_n), (y_1, ..., y_n)$ *in* $\mathbb{R}^n$ *such that* $(x_1, ..., x_n) \geq (y_1, ..., y_n)$ *and* $(x_1, ..., x_n) \leq (y_1, ..., y_n)$*. Therefore, coordinatewisely, we have both* $x_i \geq y_i$ *and* $x_i \leq y_i$ *for any* $1 \leq i \leq n$*. Therefore,* $x_i = y_i$ *for all* $1 \leq i \leq n$*. i.e.,* $(x_1, ..., x_n) = (y_1, ..., y_n)$.

Remark. *In addition to the relation* $\geq$ *on* $\mathbb{R}^n$*, usually, we also define* $>$ *and* $>>$ *as follows,*

- *We write* $(x_1, ..., x_n) > (y_1, ..., y_n)$ *if* $x_1 \geq y_1$*,...,* $x_n \geq y_n$ *and* $(x_1, ..., x_n) \neq (y_1, ..., y_n)$.
- *We write* $(x_1, ..., x_n) >> (y_1, ..., y_n)$ *if* $x_1 > y_1$*,...,* $x_n > y_n$.

With the definitions above, we define $\mathbb{R}^n_+$ *and* $\mathbb{R}^n_{++}$*, which would be the appropriate spaces for consumption bundles.*

- $\mathbb{R}^n_+ = \{(x_1, ..., x_n) \in \mathbb{R}^n : (x_1, ..., x_n) \geq 0\}$
- $\mathbb{R}^n_{++} = \{(x_1, ..., x_n) \in \mathbb{R}^n : (x_1, ..., x_n) >> 0\}$

That is, $\mathbb{R}^n_{++}$ *is the interior of* $\mathbb{R}^n_+$.

3.5 Partial and Complete Orders

Sometimes, we will assume that a rational agent is able to compare any pair of bundles. In such case, we use the word complete orders.

An relation is a *complete preorder (order)* if it is a preorder (order) and it is complete. In contrast, a relation is a *partial preorder (order)* if the relation is not necessarily complete.

Remark. *Under such definition, any complete order must be a partial order, but not vice versa.*

It is easy to see that, on $\mathbb{R}$, $\geq$ is a complete order. Now, we give an important example of complete order on $\mathbb{R}^2$.

Definition. *The lexicographic (dictionary) order* $\succeq_L$*on* $\mathbb{R}^2$ *is defined as follows:*

$$(x_1, y_1) \succeq_L (x_2, y_2)$$ [6]

if and only if one of the following holds

1. $x_1 > x_2$

2. $x_1 = x_2$ *and* $y_1 \geq y_2$

Remark. 1. *This relation is called a dictionary relation in the sense that it respect the order for a dictionary's pages: page 21 is after page 20, which is after page19, and etc. Or, we have*

$$... \succeq_L (2, 1) \succeq_L (2, 0) \succeq_L (1, 9) \succeq_L ...$$

2. *It is an evil example, which usually serves as a counter example/ an extreme case for our latter material. For instance, lexicographic preference order will not have a real-valued utility representation.*

Proposition. *The lexicographic order has the following properties.*

1. $\succeq_L$ *is a complete order.*

[6] Read as, the pair (x_1, x_2) is lexicographically greater than or equal to the pair (x_2, y_2).

2. $\succeq_L$ *extends* $\geq$ *on* $\mathbb{R}^2$*. That is, as two sets,* $\geq \leq \succeq_L$*. Or equivalently, we have*

$$(x_1, y_1) \geq (x_2, y_2) \Longrightarrow (x_1, y_1) \succeq_L (x_2, y_2)$$

Proof. To prove the first property, it is an exercise to check it is an order. To see the relation is complete, it is sufficient to check for any pairs $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, we must have either $(x_1, y_1) \succeq_L (x_2, y_2)$ or $(x_2, y_2) \succeq_L (x_1, y_1)$. We discuss case by case.

Case 1 If $x_1 > x_2$, we have $(x_1, y_1) \succeq_L (x_2, y_2)$.

Case 2 If $x_2 > x_1$, we have $(x_2, y_2) \succeq_L (x_1, y_1)$.

Case 3 If $x_1 = x_2$, we compare the second coordinate:

Case 3.1 if $y_1 \geq y_2$, we have $(x_1, y_1) \succeq_L (x_2, y_2)$.

Case 3.2 if $y_1 < y_2$, we have $(x_2, y_2) \succeq_L (x_1, y_1)$.

The second property follows directly from the definition. ■

By the above proposition, we notice that the lexicographic order extends a partial order $\geq$ on $\mathbb{R}^2$ to a complete order. Naturally, we wonder if such extension is possible for a more general class of partial orders. The answer is assertive, and given as the following theorem:

Theorem (Szipirajn)**.** *Every partial order can be extended to a complete order.*

3.6 Optima on $\mathbb{R}$

Having defined orders $\geq$ on $\mathbb{R}$, we are able to define optima (maximum, minimum, supremum, infimum) on $\mathbb{R}$. Note these definitions also work for any general orders on a set, but we will not get into there in this course.

Given a non-empty set $S \subset \mathbb{R}$, we define $x \in S$ is a *maximum of the set* S if for all $y \in S$, we have $x \geq y$. Equivalently, $x \in S$ is a maximum of the set S if and only if there does not exists a $y \in S$ such that $y > x$.

In contrast, we define $x \in S$ is a *minimum of the set* S if for all $y \in S$, we have $y \geq x$. Equivalently, $x \in S$ is a minimum of the set S if and only if there does not exists a $y \in S$ such that $y < x$.

It worth noting, the maximum (or minimum) of an infinite subset of $\mathbb{R}$ might be empty, e.g. $(0,1)$ or $(-\infty,\infty)$. However, the maximum (or minimum) of a finite subset of $\mathbb{R}$ always exists.

To deal with this nonexistence of optima, we define supremum and infimum respectively.

To start with, we say $x \in \mathbb{R}$ is an *upper (or lower) bound of* S if $x \geq y$ or $x \leq y$ for all $y \in S$.

Next, we define supremum and infimum of a set S as follows. The *supremum of a set* S is defined as to be the the least upper bound of S, whenever it exists. More rigorously:

$$\sup(S) = \begin{cases} +\infty, & \text{if there is no upper bound of } S, \\ \text{the least upper bound of } S, & \text{if there is a upper bound of } S, \\ -\infty, & \text{if } S = \emptyset. \end{cases}$$

The *infimum of a set* S is defined as to be the the largest lower bound of S, whenever it exists. More rigorously:

$$\inf(S) = \begin{cases} -\infty, & \text{if there is no lower bound of } S, \\ \text{the largest lower bound of } S, & \text{if there is a lower bound of } S, \\ +\infty, & \text{if } S = \emptyset. \end{cases}$$

We note that if a set has a maximum (or minimum), then this number has to be the supremum (or infimum) of the same set.

Sometimes, operationally, these equivalent definitions of the supremum and infimum of a set are useful:

- $\sup S = x \in \mathbb{R}$ iff x is an upper bound of S, and for all $\varepsilon > 0$, we can find a $y \in S$ such that $x - \varepsilon < y$.

- $\inf S = x \in \mathbb{R}$ iff x is an lower bound of S, and for all $\varepsilon > 0$, we can find a $y \in S$ such that $x + \varepsilon > y$.

Exercise. *It is an exercise to check these two definitions of supremum and infimum coincides for the case* $\sup S$ *and* $\inf S$ *are real numbers.*

Exercise. *To get yourself familiar with the definitions of optima, determine the maximum, minimum, supremum, infimum of the following sets:*

- $S = (-1, 1]$
- $S = (-\infty, 0)$
- $S = [0, 1] \cup [2, +\infty)$
- S *is the set of non-negative rational numbers*
- S *is the set of positive irrational numbers*[7]
- $S = \{\textit{the maximum of function } f_\sigma : \sigma \in (0, \infty)\}$, *where the real-valued function* f_σ *on* $\mathbb{R}$ *is defined by:*

$$f_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

Here, f_σ *is the density of a Gaussian distribution with a variance* σ^2.

[7] A irrational number is a number that is not rational.