Lecture 12: Preference And Utility Representation

Advanced Microeconomics I, ITAM

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There are three major ways to model choices.

- Preferences: provides a casual foundation of modeling choices, captures intuition and the characteristics of "rational" choice
- Utility function: analytically convenient
- Choice function: describes what are observed

In this notes, we will focus on the first two concepts and the relation between these two concepts.

1 Preference

We use a set X be a set of possible choices. For instance, an infinite set $X \subset \mathbb{R}^n$ could denote the budget set in consumer problem, a finite set $X \subset \{1, 2, ..., n\}$ could denote the candidate in a voting problem, and there are many other examples.

A *preference* $\succeq$ is a binary relation on X. That is, $\succeq$ is a subset of $X \times X$. For $(x, y) \in \succeq$, we write $x \succeq y$, and interpret it as x is at least as good as y.

By this definition, we could also define $x \succ y$ and $x \sim y$:

- $x \succ y$: $x \succeq y$ but $y \succeq x$ does not hold. In this case, we say x is strictly preferred to y.
- $x \sim y$: both $x \succeq y$ and $y \succeq x$ hold. In this case, we say x and y are indifferent.

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Given a preference $\succeq$, we usually impose the assumptions that $\succeq$ is complete and transitive. Sometimes, we say a preference is *rational* if it is complete and transitive. Recall the following definitions:

- $\succeq$ on X is *complete* if for all $x, y \in X$, either $x \succeq y$ or $y \succeq x$.
- $\succeq$ on X is *transitive* if for all $x, y \in X$, $x \succeq y$ and $y \succeq z$ implies $x \succeq z$.

Remark.

1. *$\geq$ on $\mathbb{R}$ is complete. $\geq$ on $\mathbb{R}^2$ is not complete when $n \geq 2$, and the lexicographic preference is a completion of this partial order.*
2. *The majority rule is not transitive. For instance:*
   - *Mom: $x \succeq y \succeq z$*
   - *Dad: $y \succeq z \succeq x$*
   - *Son: $z \succeq x \succeq y$*

   We note under majority rule, $x \succeq y$ and $y \succeq z$, but $x \succeq z$ does not hold.

2 Utility Representation

A *utility function* u is a map from X to $\mathbb{R}$. We say a utility function u *represents* a preference $\succeq$ if

$$x \succeq y \iff u(x) \geq u(y)$$

It is straight forward to notice the following proposition:

Proposition. *If u represents $\succeq$, then $\succeq$ is complete and transitive.*

Proof. Exercise. ■

Moreover, the utility representation is invariant under strictly increasing transformations.

Proposition. *If u represents $\succeq$, then any $f(u)$ represents $\succeq$ if f is a strictly increasing function on $\mathbb{R}$.*

Proof. $x \succeq y$ if and only if $u(x) \geq u(y)$ if and only if $f(u(x)) \geq f(u(y))$. ■

Corollary. *The utility representation is invariant under positive affine transformations.*

For this reason, we know it does not make direct sense to compare payoffs of different agents, or to maximize the sum of payoffs of a society.

Next, we consider the relationship between preference and utility function, in particular, what type of preference can be represented by a utility function (so we can conveniently analyze it).

First, we claim when the set of possible choices is finite, then any preference has a utility representation.

Proposition. *When X is a finite set, any preference has a utility representation.*

Proof. We prove by construction. First, we partition the set of possible choices X according to $\succeq$ recursively: First, we define the set of minimal elements in X by $X_1 \subset X$. (x_1 is minimal if for all $y \in X$, $y \succeq x_1$.) As there are only finite number of possible choices, we note X_1 is non-empty. Now, among the remaining elements in $X - X_1$, we define the set of minimal elements by X_2, and so on. Such process must stop in at most $|X|$ steps. Now, we define the utility function u by

$$u(x_k) = k, \forall x_k \in X_k$$

It is an exercise to verify that u represents $\succeq$. ■

However, we note when X is finite, our analytical/optimization tool studied previously is not very useful. Indeed, it is often the case that discrete optimization is harder than smooth optimizations. Therefore, for the ease of analysis, we usually work on an infinite set of possible choices X.

Unfortunately, when X is infinite, not all preferences have utility representations.

Proposition. *Lexicographic preferences do not have utility representation.*

Proof. We prove the case on $\mathbb{R}^2$. General cases are the same. We start by assuming the lexicographic preference has a utility representation u. Then, for any $x \in \mathbb{R}$, as $(x, 1) \succ (x, 0)$ we have $u(x, 1) > u(x, 0)$ in $\mathbb{R}$.

Define a function $q : \mathbb{R} \to \mathbb{Q}$ by

$$q(x) \in (u(x,0), u(x,1)) \cap \mathbb{Q}$$

We claim that q is an injective function. (The proof of the claim is left as an exercise.) However, there is no injective map from $\mathbb{R}$ to $\mathbb{Q}$, as $\mathbb{R}$ is uncountable while $\mathbb{Q}$ is countable. ■

Remark. *We say a set S is countable if there is an injective map from S to $\mathbb{N}$. Otherwise, the set is uncountable.*

- $\mathbb{N}$ *is countable.*
- $\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$ *is countable.*
- $\mathbb{Q}$ *is countable. To see it, we first write* $\mathbb{Q}$ *in an array*

$$\begin{array}{cccc} \frac{0}{1} & \pm\frac{1}{1} & \pm\frac{2}{1} & \ldots \\ \frac{0}{2} & \pm\frac{1}{2} & \pm\frac{2}{2} & \ldots \\ \frac{0}{3} & \pm\frac{1}{3} & \pm\frac{2}{3} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{array}$$

  We note all rational numbers are in this array, and we can count them in order by entries: (1,1), (2,1), (1,2),(3,1), (2,2), (1,3)... and omit the repeated terms. That is, we could write $\mathbb{Q}$ *in a sequence by*

$$0, 1, -1, 2, -2, \frac{1}{2}, -\frac{1}{2}, ...$$

- $\mathbb{R}$ *is not countable.*

  We prove that the interval $(0,1)$ is uncountable. Therefore, $\mathbb{R}$, containing the open interval $(0,1)$ is uncountable. To see it, we first assume that $(0,1)$ is countable. It means, we could write all numbers in $(0,1)$ as a sequence. We write them in decimal as follows:

$$x_1 = 0. * * * * * * * * *$$

$$x_2 = 0. * * * * * * * * *$$

$$x_3 = 0.*********$$

$$\vdots$$

Now, we define a new number $x \in (0,1)$, where its k-th digit is different from the k-th digit of x_k. Therefore, x is not in this sequence. Therefore, $(0,1)$ is uncountable.

3 Continuous utility representation in $\mathbb{R}^n$

The continuity of preference is usually regarded as a technical assumption that is convenient for the analysis. However, such restriction is usually non-controversial for empirical researches, as we will need an infinite data set to contradict continuity. In this section, we assume the set of possible choices $X = \mathbb{R}^n$.

Formally, we say a preference $\succeq$ is continuous if the preference is preserved at the limit: A preference $\succeq$ is *continuous* if for any sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in X, if $x_n \to x$, $y_n \to y$, and $x_n \succeq y_n$, then $x \succeq y$.

An equivalent definition of continuity of preference is that: for all $x \in X$, the better than set (or upper contour set) $\{y \in X : y \succeq x\}$ and the worse than set (or the lower contour set) $\{y \in X : x \succeq y\}$ are closed. We note this second definition of continuity implies that the indifference curves $\{y \in X : x \sim y\}$ is closed, as

$$\{y \in X : x \sim y\} = \{y \in X : y \succeq x\} \cap \{y \in X : x \succeq y\}.$$

Exercise. *For complete and transitive preference, prove the above two definitions are equivalent.*

Next, we notice that not all preferences are continuous.

Example. *Lexicographic preferences on $\mathbb{R}^2$ is not continuous: note*

$$(10^{-n}, 0) \succeq (0, 1)$$

However, at the limit $(0,0) \succeq (0,1)$ does not hold.

We notice in above example, the lexicographic preference has no utility representation. On the contrary, if a preference is continuous, then we can almost claim that it has a utility representation, and this utility representation is continuous.

Theorem (Debreu's Theorem). *For any convex set $X \subset \mathbb{R}^n$, a complete and transitive preference $\succeq$ has a continuous utility representation if and only if $\succeq$ is continuous.*

We will not prove this theorem in this class. However, you are encouraged to think or find a proof of this statement. In the following, we will only prove this theorem when the preference is monotone.

First, we say a preference $\succeq$ is *monotone* if for all $x, y \in X$, when $y >> x$, we have $y \succ x$.

Remark. *c.f. We say a preference $\succeq$ is strongly monotone if for all $x, y \in X$, when $y > x$[1], we have $y \succ x$.*

- *Any strongly monotone preference must be monotone.*
- *A preference represented by $u(x, y) = \min(x, y)$ is monotone but not strongly monotone.*
- *A preference represented by $u(x, y) = x + y$ is strongly monotone.*

Next, we state the statement we will prove.

Theorem. *Given a complete and transitive preference $\succeq$ on $\mathbb{R}^n_+$, if $\succeq$ is continuous and monotone, then it has a continuous utility representation.*

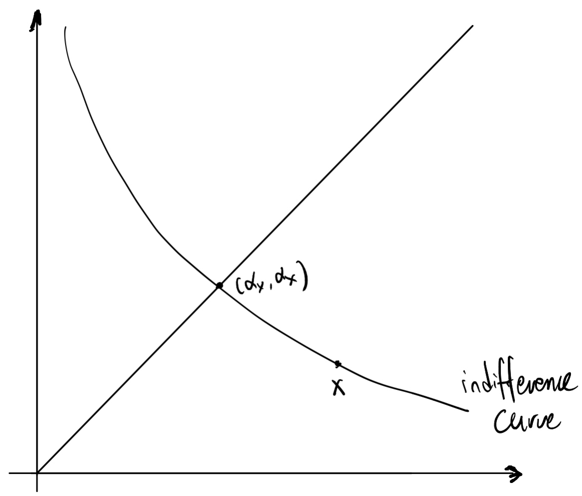
Remark. *In this statement, the monotonicity can be replaced by locally non-satiated: we say $\succ$ is locally non-satiated if for all $x \in X$, and small number $\varepsilon > 0$, there is a $y \in X$ such that $y \succ x$.*

Proof. In this proof, we will first state an observation and use this observation to finish the proof. Then, we justify the observation we stated at the beginning.

Claim: for any $x \in \mathbb{R}^n_+$, there is a unique $\alpha_x \in [0, \infty)$ such that

$$x \sim (\alpha_x, \alpha_x, ..., \alpha_x)$$

[1] Again, it means $y \geq x$ and $y \neq x$.



Now, assuming the claim is correct, we define a utility function by

$$u(x) = \alpha_x$$

Now, we need to show u is continuous and represents the preference.

To see u is continuous, we take $x_n \to x$ in X. By the claim, we have

$$x_n \sim (\alpha_n, \alpha_n, ..., \alpha_n)$$

$$x \sim (\alpha, \alpha, ..., \alpha)$$

for some α_n and α in $[0, \infty)$. We wish to show $\alpha_n \to \alpha$. We prove by contradiction. Assume it is not true, by selecting subsequences, we have $\alpha_n > \alpha + \varepsilon$ for some ε. (Or the symmetric case $\alpha_n < \alpha - \varepsilon$.) By monotonicity, we note x_n is in the better than set of $(\alpha + \varepsilon, \alpha + \varepsilon, ..., \alpha + \varepsilon)$. By continuity, this better than set is closed. Therefore, x is in the better than set of $(\alpha + \varepsilon, \alpha + \varepsilon, ..., \alpha + \varepsilon)$. By transitivity, $(\alpha, \alpha, ..., \alpha)$ is in the better than set of $(\alpha + \varepsilon, \alpha + \varepsilon, ..., \alpha + \varepsilon)$, which contradicts monotonicity.

To see u represents $\succeq$:

$$\begin{aligned} x \succeq y &\iff (\alpha_x, \alpha_x, ..., \alpha_x) \succeq (\alpha_y, \alpha_y, ..., \alpha_y) \text{ by transitivity} \\ &\iff \alpha_x \geq \alpha_y \text{ by monotonicity} \\ &\iff u(x) \geq u(y) \text{ by definition} \end{aligned}$$

Last, we see the claim is correct: for any $x \in \mathbb{R}^n_+$, we check the following two sets:

$$\{\alpha \in \mathbb{R}_+ : (\alpha, \alpha, ..., \alpha) \succeq x\}$$

$$\{\alpha \in \mathbb{R}_+ : x \in (\alpha, \alpha, ..., \alpha)\}$$

We have the following observation.

1. Both sets are non-empty. (For the first, we take $\alpha = \max_k x_k$. For the second, we take $\alpha = 0$. By monotonicity, we know these sets are non-empty.)

2. Both sets are closed, as $\succeq$ is continuous.

3. Their union is $\mathbb{R}_+$, by the completeness.

Therefore, their intersection must be non-empty. (Why?) That is, there is a $\alpha_x \in \mathbb{R}_+$ such that

$$x \sim (\alpha_x, \alpha_x, ..., \alpha_x)$$

To see this intersection is a singleton, we leave it as a simple exercise. ■

4 Convexity of preferences in $\mathbb{R}^n$

We say a preference $\succeq$ is *(strictly) convex* if for all $x \in X$, the better than set $\{y \in X : y \succeq x\}$ is (strictly) convex.

Intuitively, when a preference is convex, it means agents prefers diversity: taking $x \sim y$, we have $\frac{1}{2}x + \frac{1}{2}y \succeq x \sim y$.

Depending on the context, this assumption may not make sense. For instance, you may like beer or wine, but it is rare you will enjoy a mixture of beer and wine.