Starting today, we will study the classical general equilibrium theory. I will follow Gerard Debreu’s monograph *Theory of Value* as much as I can. Some additional interesting topics in the following reference might be mentioned:

- Werner Hildenbrand, *Core and Equilibria of a Large Economy*, 1974

Throughout this lecture, we will focus on pure exchange market. That is, an economy with only consumers and there is no production factor. For extensions to economy with productions, please consult Debreu’s monograph. Moreover, we will just focus on understanding the basic model. For this reason, we will only deal with an economy with agents who have strictly concave utility functions 1.

In this part, first, we will develop the model - (Walrasian / Competitive) Economy, and understand what is prices (resulting from the interaction of the agents through markets) and the role of prices in an optimal state of an economy. More precisely, we will try to understand the existence and the optimality of competitive equilibria. The proof of the existence result is a highlight in the development of this subject, and the optimality of competitive equilibria formalizes Adam Smith’s intuition on the invisible hand of the market.

Then, we will go to the cooperative part of the model from the viewpoint of Edgeworth. By introducing the solution concept core, which is an economic state that no group of agents can jointly do better than the given state, we will ask ourselves again what is the meaning of prices and competitions.

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*Please email me at xinyang.wang@itam.mx for typos or mistakes. Version: April 6, 2021.
1That is, demand maps are functions.
Depending on the time, I may or may not talk about the following topics:

- the dynamic of prices - the tâtonnement stability of equilibrium: if prices change according to the excess demand (i.e. if more people want to have a commodity, the price of this commodity increases and vice versa), whether or not prices will converge to the equilibrium price.

- the general equilibrium models over time - the Arrow-Debreu model and the general equilibrium model with a incomplete market: in a dynamic model with multiple dates, given complete information, how agents will trade assets with uncertainty with each other and what would be the price of these assets.

- the general equilibrium model with unit demand: Koopmans’ matching model in the transferable utility case and Gale-Shapley algorithm in the non-transferable utility case: given a distribution of resources, suppose we know distribution of the demand of these resources, what is the optimal (cost minimized) way to pair sellers and buyers and what would be the profit/price these agents pay.

1 History

In this course, we will focus on the general equilibrium model developed in the 1950s and 1960s initiated by Kenneth Arrow, Gerard Debreu, Lionel McKenzie, and developed by many others. For the history of this subject, I quoted a paragraph from the preface of McKenzie's book. I suggest you read the whole preface.

General equilibrium theory in the modern sense was first developed in the second half of the nineteenth century by Francis Edgeworth, Alfred Marshall, and León Walras, most systematically by Walras. In the first half of the century some earlier moves in the direction of formal analysis of competitive markets using mathematics had been made by Augustin Cournot and Jules Dupuit. Then in the early twentieth century Vilfredo Pareto and Gustav Cassel added some additional formulations to this theory. However, the modern elaboration and rigorous development of general equilibrium theory from these foundations was begun in the 1930s and 1940s by John Hicks and Paul Samuelson, in the tradition of academic economics but with liberal appeal to mathematics, and by Abraham Wald and John von Neumann, from a rigorous mathematical viewpoint. Frank Ramsey in the late 1920s and von Neumann in the 1930s had laid the ground for optimal growth theory, which I relate to general equilibrium over time. However, the
general equilibrium theory that this book is concerned to present was developed in the second half of the twentieth century primarily by Kenneth Arrow, Gerard Debreu and me but with many contributions from others. In particular, Tjalling Koopmans should be mentioned for his activity analysis and optimal growth theory....

2 Features of the general equilibrium models

Before we get in to the model, we first spend some time on some features of the general equilibrium models: agent optimization, market clearing and perfect competition.

1. Agent optimization:

- agents are supposed not to act mechanically. e.g. repeating actions from the past no matter how circumstance changes, or following simple rule of thumbs.
- Instead, it is supposed that they hold their objectives constantly in mind, adapting their actions when circumstance changes.

For this reason, the Walrasian tradition lies in the heart of an individualistic political philosophy by reducing aggregate action to the sum of individual actions motivated by individual objectives, then evaluating social success by aggregating individual opinions of personal success.

2. Market Clearing:

- the steady state, equilibrium, of the market is characterized to be the state that the total supply and the total demand are equivalent for all commodities.

3. Perfect Competition:

- Implicitly, there are a large number of agents in the market.
- Each agent is identical to a large number of other agents.
- Agents take price as given.
- Agents only care about their consumption, and do not care about with whom he is trading with (trade is anonymous) or the happiness of the other agents.
3 Exchange Economy

We study an economy with $I$ agents and $N$ commodities. Thus, the set of consumption bundles is $\mathbb{R}^N_+$. An element $x = (x_1, ..., x_n) \in \mathbb{R}^N_+$ is a consumption bundle.

We abuse the notation $I$ by using it as both an index set of agents and the cardinality of this set. Each agent $i \in I$ is described by his preference $\succeq_i$ over the consumption bundles $\mathbb{R}^N_+$ and his endowment $\omega_i \in \mathbb{R}^N_+$.

- We will assume $\succeq_i$ is locally non-satiated and is continuous.
- Thus, by Debreu’s theorem, $\succeq_i$ can be represented by a continuous function $u_i$ on the set of consumption bundles.
- For the cleanness of our analysis, we will assume $u_i$ to be strictly concave.$^2$

An *Walrasian/ exchange economy* is characterized by a tuple $\mathcal{E} = (I, (u_i, \omega_i)_{i \in I})$.

4 Competitive Equilibrium

The key of competitive equilibrium is that we wish to use market price to understand a stable state of the competitive market.

Formally, a *price vector/ prices* are defined to be an nonzero element in $\mathbb{R}^N_+$:

$$p = (p_1, ..., p_N) \in \mathbb{R}^N_+ - \{0\}$$

In particular, $p_n$ is the price of commodity $n$.

Facing the same price $p$, agents, given their endowments, they compute their optimal

$^2$We note this assumption rules out some interesting preference, for instance, the preference over indivisible goods, or risk loving preference. However, we will need this assumption for the demand map to be a function. (That is, the solution of individual maximization problem has a unique solution.) To get rid of this technical assumptions, we will need to work on a continuum of agents set, that is, $I = [0, 1]$. This setting precisely coincides with our understanding of the competitive market that every agent has a zero market power, and the Lyapunov theorem asserts the aggregate better than set is convex, which is enough for our analysis. However, this discussion is way beyond the scope of this lecture.
consumption bundle within the budget set. Mathematically, agent $i$ solves

$$\max_{y \in B(p, \omega_i)} u_i(y)$$

where the budget set of agent $i$ is defined to be

$$B(p, \omega_i) = \{ y \in \mathbb{R}^N_+ : p \cdot y \leq p \cdot \omega_i \}$$

Given prices $p$, we can ask if the market can reallocate resources such that every agent consumes their optimal consumption bundle in their budget set. Suppose that there is a Walrasian auctioneer that can manipulate the market prices as he wishes, we wonder would there be prices such that the market can satisfy all agents by some reallocation of the initial resources. That is, whether by trading under some prices, the outcome of trade is optimal for all agents. Such an individually optimal state is called a competitive equilibrium.

Formally, an allocation is a tuple of consumption bundles for every agents:

$$x = (x_1, ..., x_I) = (x_i)_{i \in I} \in \mathbb{R}^{NI}_+$$

We say an allocation $x$ is feasible if it can be obtained by some reallocation/trade of the initial resources. i.e.

$$\sum_{i \in I} x_i = \sum_{i \in I} \omega_i$$

This equation is usually referred to as the market clearing condition, as the left hand side of the equality is the total demand of the resources, and the right hand side of the equality is the total supply/endowment of the resources. We note, this equation is about vectors. So, if we look at the $n$-th row of the equation, it implies the market clearing in the market of every commodities:

$$\sum_{i \in I} (x_i)_n = \sum_{i \in I} (\omega_i)_n$$

A pair $(p, (x_i)_{i \in I})$ is a competitive equilibrium of economy $\mathcal{E}$ if

- $p$ is a price
• \((x_i)_{i \in I}\) is a feasible allocation. i.e. the market clearing condition holds

\[
\sum_{i \in I} x_i = \sum_{i \in I} \omega_i
\]

• Individual optimality: \(x_i\) maximizes agent \(i\)'s payoff among the budget set:

\[
u_i(x_i) \geq u_i(y_i), \quad \forall y_i \in B(p, \omega_i)
\]

Some properties of the competitive equilibrium is immediate:

**Proposition** (Normalization of prices). *For any competitive equilibrium \((p, (x_i)_{i \in I})\) of economy \(E\), \((\lambda p, (x_i)_{i \in I})\) is a competitive equilibrium, for all \(\lambda > 0\).*

That is, we are free to scale up and down the equilibrium prices. For this reason, it provides us one free dimension for determining the prices. Some usual normalization of prices are given as follows:

• One price can be normalized to be 1: \(p_1 = 1\). Such commodity, in this case commodity 1, is called a numeraire.

• The sum of prices is normalized to be 1: \(p_1 + ... + p_N = 1\). That is, the set of prices lie on a simplex.

• The square sum of prices is normalized to be 1: \(p_1^2 + ... + p_N^2 = 1\). That is, the set of prices lie on a unit ball.

Each normalization has its own advantage. We may encounter some types of normalizations.

**Proof.** Scaling up or down the prices has no effect on the budget set: \(B(p, \omega_i) = B(\lambda p, \omega_i)\).  

**Proposition** (Walras’ Law). *For any competitive equilibrium \((p, (x_i)_{i \in I})\) of economy \(E\),

\[
p \cdot x_i = p \cdot \omega_i, \forall i \in I
\]

**Proof.** This proposition suggests that agents are consuming all their incomes, which is by the local nonsatiation of preferences.
**Corollary.** When the prices is interior, one of the market clearing conditions is redundant.

**Proof.** If the market clearing condition holds for all markets besides the market of commodity $c$:

$$\sum_{i \in I} (x_i)_n = \sum_{i \in I} (\omega_i)_n, \forall n \neq c$$

By Walras’ law,

$$p \cdot x_i = p \cdot \omega_i, \forall i \in I$$

Therefore,

$$\sum_{i \in I} p \cdot x_i = \sum_{i \in I} p \cdot \omega_i$$

$$\iff \sum_{i \in I} \sum_{n=1}^N p_n \cdot (x_i)_n = \sum_{i \in I} \sum_{n=1}^N p_n \cdot (\omega_i)_n$$

$$\iff \sum_{i \in I} \sum_{n \neq c} p_n \cdot (x_i)_n + p_c \cdot (x_i)_c = \sum_{i \in I} \sum_{n \neq c} p_n \cdot (\omega_i)_n + p_c \cdot (\omega_i)_c$$

By the market clearing condition of markets other than commodity $c$, the first summation on the left hand side and the right hand side coincides, therefore,

$$\sum_{i \in I} p_c \cdot (x_i)_c = \sum_{i \in I} p_c \cdot (\omega_i)_c$$

As $p_c > 0$, we have

$$\sum_{i \in I} (x_i)_c = \sum_{i \in I} (\omega_i)_c$$

That is, the market is clear in commodity $c$'s market. □
5 Edgeworth Box

5.1 Edgeworth Box

In this section, we analyze the Edgeworth economy: an economy with 2 agents and 2 commodities, and we introduce Edgeworth box to visualize the competitive equilibrium and see the mathematical property of equilibrium prices.

Consider an economy with two agents, agent A and agent B, and two commodities, commodity 1 and commodity 2. We have already talked about how to determine the demand for a single agent - the point that budget line tangent to the indifference curves. Our first obstacle is to represent the whole economy by using one graph. One way to do so is drawing the Edgeworth box:

![Figure 1: The Edgeworth Box](image)

In a Edgeworth box, the are two axises of the two agents. Agent 1’s axis is in the usual position and agent 2’s axis is upside down, with its original given by the coordinate of \((\omega_A + \omega_B)\). The endowments of two agents is represented by a single point \(\omega\) in the diagram - evaluated in axis 1, it is \((\omega_{A1}, \omega_{A2})\), and evaluated in axis 2, it is \((\omega_{B1}, \omega_{B2})\). Arranged in this way, the set of points in the box corresponds to the set of feasible allocations.

5.2 Budget Sets

Next, we want to represent the budget sets of these two agents under some price. We note that for the budget set of agent A, his budget set is \(B(p, \omega_A) = \{x \geq 0 : p \cdot x \leq p \cdot \omega_A\}\), which
can be represented by the region below a budget line (red region), which passes through the endowment point and has a normal vector $p = (p_1, p_2)$. It is a helpful coincidence that the region above the budget line (blue region) is agent B’s budget set, viewed in axis 2. (Prove it.)

![Figure 2: Budget sets in an Edgeworth box](image)

### 5.3 Equilibrium Allocation

We note a point (an allocation) in the Edgeworth box is a Walrasian allocation if at this point, the indifference curves of two agents are tangent to each other. View this observation in a different way, we know that a point, which corresponds to an allocation, is a walarasian allocation if 1) two indifference curves tangent to each other at this point, 2) the separating hyperplane of the two better than sets at this point passing through the endowment.

![Figure 3: Equilibrium allocation in an Edgeworth box](image)

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3Again, an competitive equilibrium is a pair consisting an allocation and a price.
5.4 Offer Curves

Now, we see how we could use offer curves to determine the set of competitive equilibria. First, we see what are offer curves.

Recall that offer curve of agent $i$ is the demand curve of this agent

$$\xi_i(p) = \{x_i : p \cdot x_i \leq p \cdot \omega_i\}$$

Note, the budget lines in an Edgeworth box rotate around the initial endowment $\omega$, as the ratio $p_2/p_1$ changes\(^4\). For each budget sets, we can find a demand, and the offer curve is just the curve connecting these demands.

![Offer Curve](image)

Figure 4: An offer curve

Here are some observations on the offer curves.

**Remark.**

1. *Offer Curves always pass through the endowment point.* (As there is an indifference curve passing through the endowment, and its tangent line at the endowment correspond to a budget line.)

2. As $\omega_i$ is always affordable, every point on the offer curve is better than $\omega_i$.

3. At equilibrium, the offer curves of two agents intersect.

\(^4\)Here, only the ratio of prices matter, as we can normalize one price, $p_1$, to be 1
• Any intersection of the offer curves outside $\omega$ correspond to a competitive equilibrium.

4. Equilibrium may not be unique.

Now, we give some examples of offer curves.

**Example 1.** Let $u_A = x_1^{1/4}x_2^{1/2}$, $\omega_A = (6, 0)$. As the utility function is Cobb-Douglas, we know agent will spend a proportion of his income on commodity 1 and 2, and the proportion is given by the coefficient of the Cobb-Douglas function. That is,

$$p_1x_1^* = \frac{1/4}{1/4 + 1/2}w = \frac{1}{3}w = \frac{1}{3}p_1 * 6$$

We have $x_1^* = 2$. Therefore, the offer curve is given by:

**Example 2.** Let $u_A = \min(x_1, 2x_2)$, $\omega_A = (2, 3)$, the offer curve is given by:
Example 3. Let $u_A = 2x_1 + x_2$, $\omega_A = (1, 1)$, the offer curve is given by:

\[
\begin{align*}
\gamma_A(p) : [0, \infty] &\to \mathbb{R}_+^2 \\
\end{align*}
\]

5.5 Determination of Competitive Equilibrium

Next, we present the strategy for solving a Walrasian equilibrium for an Edgeworth economy. Note in the two goods case, we can normalize the price of good 1 to be 1. Therefore, the set of prices is parametrized by $(1, p)$, where $p \in [0, \infty]$.

Step 1 Write out offer curves as functions of prices

\[
x_A(p), x_B(p) : [0, \infty] \to \mathbb{R}_+^2
\]

Step 2 Market clearing condition:

\[
x_A(p) + x_B(p) = \omega_A + \omega_B
\]
Step 3 Solve $p$.

We note that this algorithm is only nice when there are two goods, as by our previous understanding, there is only one unknown parameter $p$ for the prices, and only one effective market clearing condition (as one of them is redundant). For an economy with more than two goods, we will have more than one variables and more than one market clearing conditions. As the market clearing conditions are highly nonlinear, it is very hard to compute the competitive equilibrium (except for some special cases).

**Example 4.** Let $u_A = x_1^{1/4} x_2^{1/2}$, $\omega_A = (6, 0)$, $u_B = x_1^{1/2} x_2$, $\omega_B = (0, 12)$. By the previous argument, we know $x_{A1}^* = 2$, $x_{B2}^* = \frac{1}{1+1/2} \times 12 = 8$. Therefore,

$$x_A(1, p) = (2, 4/p), x_B(1, p) = (4p, 8)$$

By the market clearing condition, we have

$$2 + 4p = 6$$
$$4/p + 8 = 12$$

Therefore, we get $p = 1$. And the competitive equilibrium is given by

$$((1, p), (x_A, x_B)) = ((1, 1), ((2), (4, 8)))$$