Lecture 17: Existence of Walrasian Equilibria

Advanced Microeconomics I, ITAM

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In this lecture, we prove the existence of a competitive equilibrium. We will first study the existence in an Edgeworth economy with two goods and two agents, and then study the existence for general models. As, in this course, we do not deal with multi-valued functions (sometimes called correspondences), the proof we will give will be based on Brouwer's fixed point theorem for functions, rather than Kakutani's fixed point theorem for correspondences. For this reason, the existence proof we give here does not directly replicate the idea used in 2 goods case. However, it saves us from the heavy lifting work of understanding (the continuity) of correspondences.

1 Existence proof for the Edgeworth Economy

In an economy with two goods and two agents, from the last lecture, we have seen that the equilibrium allocation is given by the intersection of two offer curves in an Edgeworth box, and the equilibrium price can be determined by solving the nonlinear equation specified by the market clearing condition. In general, a non-linear equation may not have a solution. For instance,

$$x^2 + 1 = 0$$

However, the economic property our demand functions have will rule out such cases.

There are numerous ways to prove the existence of a competitive equilibrium. One of them uses the idea of tatonnement process: imagine that there is a Walrasian auctioneer controlling the prices. When too many agents want to buy a commodity, he increases the

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price of this commodity, and when almost nobody wants to buy a commodity, he decreases the property. His goal is to set the prices such that the total demand is equal to the total supply, in this scenario, the total endowment.

To formalize this idea, recall that prices are normalized to $(1, p)$, where $p \in [0, \infty]$. *Individual demand (function)* of agent i is defined by

$$\xi_i(1,p) = \underset{x \in B(p,\omega_i)}{\operatorname{argmax}} u_i(x) \in \mathbb{R}^2_+$$

Aggregate demand (function) is the sum of all individual demands:

$$\xi(1,p) = \sum_{i \in I} \underset{x \in B(p,\omega_i)}{\operatorname{argmax}} u_i(x) \in \mathbb{R}^2_+$$

Excess demand (function) is the difference between aggregate demand and aggregate supply:

$$z(1,p) = \xi(1,p) - \sum_{i \in I} \omega_i = \sum_{i \in I} (\xi_i(1,p) - \omega_i)$$

The latter summand is called the excess demand (function) of agent i. Note, $z(1,p)$ is a 2-vector, we write it coordinatewisely by

$$z(1,p) = (z_1(1,p), z_2(1,p))$$

where $z_n(1,p)$ is the excess demand of commodity n.

We will focus on the property of the excess demand function $z : [0, \infty] \to \mathbb{R}^2_+$ hereafter. We start with a few observations on z:

Remark.

- *z is a function, as we assumed utility functions are strictly concave.*
- *z is continuous, as demand functions are continuous by Berge's maximization theorem.*
- *z is homogeneous of degree 0:*

$$z(\lambda, \lambda p) = z(1,p), \forall \lambda > 0$$

- *Walras' law: as for all agent i,* $(1,p)\cdot\xi_i(1,p) = (1,p)\cdot\omega_i$

$$(1,p)\cdot z(1,p) = 0$$

Now, we are ready to go to the existence result in the Edgeworth economy:

Theorem. *For an Edgeworth economy $\mathcal{E}$, if utility functions are strictly concave, continuous and strictly monotonic, there is a competitive equilibrium.*

We will prove that there is a $p \in [0,\infty]$ such that $z(1,p) = 0$. The equilibrium allocation is determined by the corresponding demand under prices $(1,p)$.

Proof. By the Walras' Law, we have

$$z_1(1,p) + pz_2(1,p) = 0, \forall p \in [0,\infty]$$

Therefore, for any positive p, we have

- When $z_1(1,p) > 0$, $z_2(1,p) < 0$.
- When $z_2(1,p) > 0$, $z_1(1,p) < 0$.
- When $z_2(1,p) = 0$, $z_1(1,p) = 0$.

Therefore, to find a p such that $z(1,p) = 0$, we just need to find a p such that $z_2(1,p) = 0$.

Note, we will use the strictly monotonicity of utility functions to claim that

$$\lim_{p\to 0} z_2(1,p) = +\infty$$

$$\lim_{p\to\infty} z_1(1,p) = \lim_{p\to\infty} z_1(1/p,1) = +\infty$$

We note this claim is intuitively reasonable: when a commodity becomes arbitrarily cheap, both agents would buy more and more of it.

To see its validity, formally, we will prove the first result. The proof of the second result is identical. We prove by contradiction. Suppose that $z_2(1,p)$ is bounded as $p \to 0$, there is a subsequence p_n such that $p_n \to 0$ and

$$z_2(1, p) \to x$$

for some $x \in \mathbb{R}$. By the continuity of excess demand z, $z_2(1, 0) = x$, which means when commodity 2 is costless, both agents will just consume a finite amount of it, which contradicts strict monotonicity, as more consumption even on one good is always preferred.

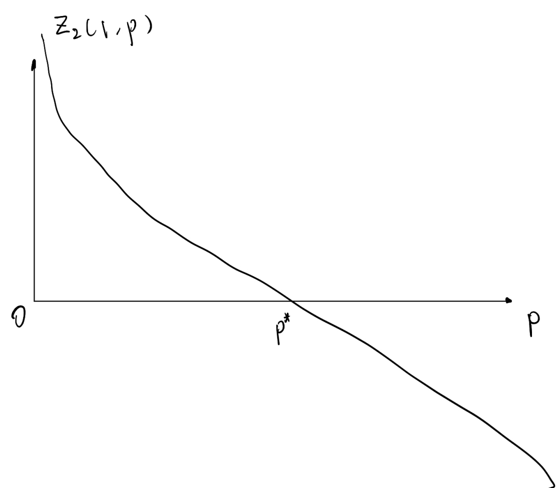
Having this claim, by the Walras' law, we have

$$\lim_{p\to\infty} z_2(1, p) < 0$$

as $\lim_{p\to\infty} z_1(1, p) > 0$. Together with claim that

$$\lim_{p\to 0} z_2(1, p) > 0$$

By intermediate value theorem, there is a p^* such that $z_2(1, p^*) = 0$.



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2 Existence Proof for General Economies

Now, we will give an existence result of competitive equilibrium, and develop a proof for a slightly weaker case. That we only give the proof for a slightly weaker case is to minimize the level of technicalities and to highlight the key idea. However, we will give all essential ideas.

Theorem 1. *For an economy $\mathcal{E} = (I, (u_i, \omega_i)_{i \in I})$, if*

- *(A1) for all $i \in I$, u_i is locally non-satiated.*
- *(A2) for all $i \in I$, u_i is continuous.*
- *(A3) for all $i \in I$, u_i is (quasi-)concave.*
- *(A4) $\sum_{i \in I} \omega_i >> 0$ and $\omega_i > 0$ for all $i \in I$.*

Then, a competitive equilibrium of $\mathcal{E}$ exists.

The first three assumptions $(A1) - (A3)$ need no explanations. Assumption $(A4)$ states that no good is trivial (have a zero amount in the market) and no agent is trivial (have a zero resource to trade with the others). We remark again that Assumption $(A3)$ is a technical assumption that is only needed for the existence of a finite economy (when I is finite). This assumption rules out the possibility of externality and indivisible goods.

The proof we will develop hereafter requires a set of slightly stronger assumptions:

- (A1') for all $i \in I$, u_i is strictly monotone: $x > y$ implies $u_i(x) > u_i(y)$.
- (A2) for all $i \in I$, u_i is continuous.
- (A3') for all $i \in I$, u_i is strictly concave.
- (A4') $\omega_i >> 0$ for all $i \in I$.

Before we start to develop the proof, we recall some useful observations.

1. prices are normalized such that

$$p_1 + ... + p_N = 1$$

The set of prices is denoted by P. That is, P is the n-dimensional simplex

$$P = \{p \in \mathbb{R}_+^n : p_1 + ... + p_N = 1\}$$

2. demand functions $\xi_i(p) : P \to \mathbb{R}_+^N$ are continuous.

3. excess demand function $z : P \to \mathbb{R}_+^N$ defined by

$$z(p) = \sum_{i \in I} (\xi_i(p) - \omega_i)$$

   is continuous.

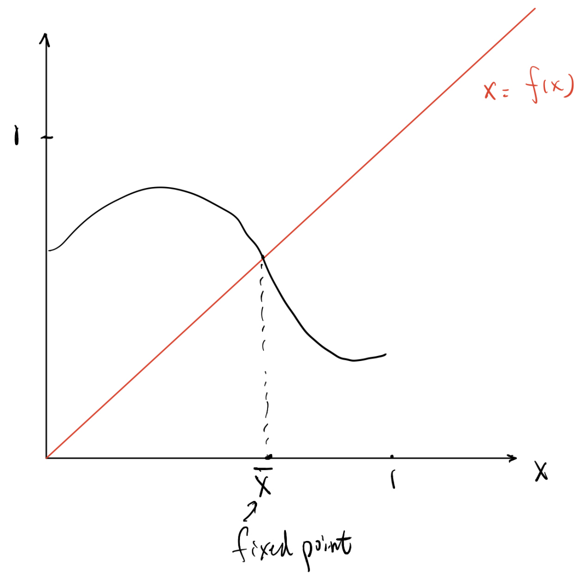
4. the excess demand function z satisfies the Walras' law

$$p \cdot z(p) = 0, \forall p \in P$$

2.1 Brouwer's fixed point theorem

We need some heavy lifting tool to understand the classical proof. In previous lectures, we noticed that the optimization techniques are crucial to understand the behavior of a rational agent, no matter he is a consumer, a producer or a decision maker facing uncertainty. For models with multiple agents, economists usually focus on the concept equilibrium, and the crucial related technique is the fixed point theorem. Now, we present Brouwer's fixed point theorem.

For a function $f : X \to X$, a *fixed point* of f is a point $\bar{x} \in X$ such that $\bar{x} = f(\bar{x})$.



Theorem (Brouwer's fixed point theorem)**.** *For a function $f : X \to X$, if f is continuous and X is non-empty, convex and compact*[1] *in $\mathbb{R}^N$, then f has a fixed point.*

Example.

- *X must be closed: take $X = (0, 1]$ and $f(x) = x/2$, f has no fixed point.*
- *X must be bounded: take $X = [0, \infty)$ and $f(x) = x + 1$, f has no fixed point.*
- *X must be convex: take $X = \{x \in \mathbb{R}^2 | \, \|x\| = 1\}$ and $f(x) = -x$, f has no fixed point.*
- *f must be continuous: take $X = [0, 1]$ and*

$$f(x) = \begin{cases} 1 & 0 < x \le 0.5 \\ 0 & 0.5 < x \le 1 \end{cases},$$

  f has no fixed point.

2.2 Gale-Debreu-Nikaido Lemma

The key step in proving the existence of a competitive equilibrium is the Gale-Debreu-Nikaido Lemma, which states an equilibrium exists with free disposal.

[1] Recall that the compactness in Euclidean spaces is equivalent to closedness and boundedness.

We will present Debreu's proof in 1959 using Brouwer's fixed point theorem. Clearly, we restrict ourselves to demand functions, rather than demand correspondences. (Thus we have to impose the strict concavity of utility functions.) If we are comfortable to deal with correspondence, a more intuitive proof using Kakutani's fixed point theorem for correspondences can be composed, and this intuitive proof will, on the high level, using the idea of the tatonnement process we mentioned in our two goods case.

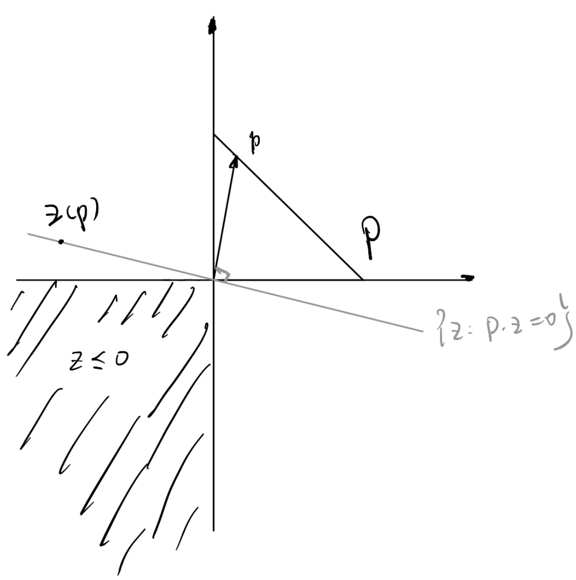
Stunningly, Uzawa (1962) observed that the Gale-Debreu-Nikaido lemma is equivalent to Brouwer's fixed point theorem. In the following, we will use Brouwer's fixed point theorem to prove the Gale-Debreu-Nikaido lemma. For the proof of the converse direction, please check Uzawa's paper.

Theorem (Gale-Debreu-Nikaido lemma)**.** *Let* $z : P \to \mathbb{R}^N$ *be a function such that*

- $p \cdot z(p) = 0$ *for all* $p \in P$
- z *is a continuous function.*

Then, there is a $\bar{p} \in P$ *such that* $z(\bar{p}) \leq 0$.

As the following figure suggests, this lemma states that, if we pick $z(p)$ continuously in p on the hyperplane passing through the origin and perpendicular to p, the curve of $z(p)$ must pass the shaded area.



Remark.

- *When $p >> 0$, the excess demand map satisfies these two conditions. When $p_n = 0$, the excess demand map is undefined. Thus, in the complete proof in the following subsection, one of our obstacle is to give a definition of excess demand when some price is zero.*

- *$z_n(\bar{p}) < 0$ only if $p_n = 0$. i.e. some market is not clear only if the corresponding price is zero. To see it, we use the first condition and obtain $\bar{p} \cdot z(\bar{p}) = 0$. That is, $\sum_n \bar{p}_n z_n(\bar{p}) = 0$. As $z(\bar{p}) \leq 0$, we have $\bar{p}_n z_n(\bar{p}) = 0$ for all commodity n.*

Proof. We start by defining a vector $g(p) \in \mathbb{R}_+^N$ for each prices $p \in P$:

$$g_n(p) = \max(0, p_n + z_n(p))$$

Clearly, $g_n(p) \geq 0$ and now we see $g(p) \neq 0$. i.e. not all of them are zero:

$$p \cdot g(p) \geq p \cdot (p + z(p)) = p \cdot p + p \cdot z(p) = p \cdot p > 0$$

Now, we define a map $\varphi : P \to P$ by

$$\varphi(p) = \frac{g(p)}{\sum_{n=1}^N g_n(p)}$$

Note this function is well-defined as $\sum_{n=1}^N g_n(p) > 0$ for all $p \in P$.

Note g as a function from P to $\mathbb{R}_+^N$ is continuous, as z is continuous. So, φ is continuous. It is instant that the price simplex P is non-empty, compact and convex. Therefore, by Brouwer's fixed point theorem, there is a $\bar{p}$ such that

$$\bar{p} = \frac{1}{\sum_{n=1}^N g_n(\bar{p})} g(\bar{p})$$

We write it as

$$\bar{p} = \lambda g(\bar{p})$$

where $\lambda = \frac{1}{\sum_{n=1}^{N} g_n(\bar{p})} > 0$. i.e.

$$\bar{p}_n = \lambda \max(0, \bar{p}_n + z_n(\bar{p})), \forall n$$

Now, we claim $\lambda = 1$: observe that $\bar{p}_n(\max(0, \bar{p}_n + z_n(\bar{p}))) = \bar{p}_n(\bar{p}_n + z_n(\bar{p}))$, as no matter $\bar{p}_n + z_n(\bar{p})$ is larger or smaller than zero, the equality always holds. Therefore,

$$\begin{aligned}
\bar{p} \cdot g(\bar{p}) &= \sum_{n=1}^{N} \bar{p}_n(\max(0, \bar{p}_n + z_n(\bar{p}))) \\
&= \sum_{n=1}^{N} \bar{p}_n(\bar{p}_n + z_n(\bar{p})) \\
&= \bar{p} \cdot (\bar{p} + z(\bar{p})) \\
&= \bar{p} \cdot \bar{p} + \bar{p} \cdot z(\bar{p}) \\
&= \bar{p} \cdot \bar{p} \\
&= \bar{p} \cdot \lambda g(\bar{p})
\end{aligned}$$

The last inequality is by the fixed point theorem. Therefore, we have $\lambda = 1$, or

$$\bar{p}_n = \max(0, \bar{p}_n + z_n(\bar{p})), \forall n$$

This equality is impossible if $z_n(\bar{p}) > 0$. So, we have $z(\bar{p}) \leq 0$. ■

2.3 Complete Proof

There are two difficulties in using the Gale-Debreu-Nikaido lemma to argue the existence of a competitive equilibrium. First, the excess demand may not be well-defined when some price is zero, so the Gale-Debreu-Nikaido lemma can not be applied directly to the excess demand map. Second, we may not have market clearing condition for some markets in which the price is zero. Thus, in the following proof, we deal with these two difficulties.

We note that due to the fact Brouwer's fixed point theorem requires a compact domain of function, we have to deal with the case that some price is zero. That is, the Gale-Debreu-Nikaido lemma is insufficient even if we impose the expost assumption that equilibrium prices are interior.

The key idea in dealing with zero price problem is Debreu's box trick: We define the demand when some price is zero by bounding the consumption set by a large box. The box is so large that the bound on each direction is larger than the maximum resources in the society. Therefore, this additional constraint will never be binding at an equilibrium, due to the market clearing condition.

Proof.

Step 1: Debreu's Box trick:

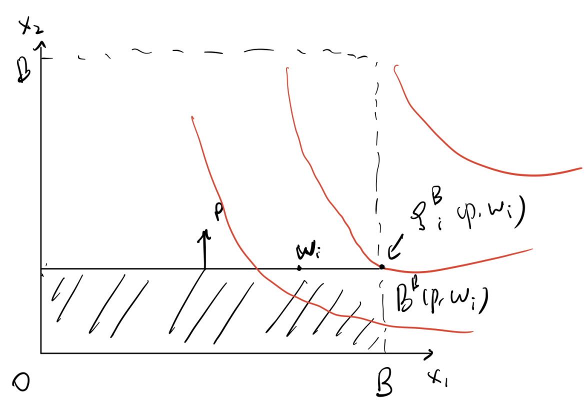
Take a large number $B > 0$ such that $B > \max_n \sum_{i \in I} (\omega_i)_n$. The restricted budget sets are in the form

$$B^B(p, \omega_i) = \{x \in \mathbb{R}^N_+ : p \cdot x \leq p \cdot \omega_i, x_n \leq B, \forall n\}$$

and restricted demand functions $\xi_i^B : P \to \mathbb{R}^N_+$ are defined by

$$\xi_i^B(p) = \underset{x \in B^B(p,\omega_i)}{\text{argmax}} \; u_i(x)$$

Note, as for all prices $p \in P$, the restricted budget sets $B^B(p, \omega_i)$ are compact. So, the restricted demand function is well-defined everywhere.



Step 2: Property of restricted excess demand:

Define restricted excess demand by

$$z^B(p) = \sum_{i \in I} (\xi_i^B(p) - \omega_i)$$

By Berge's theorem, $\xi_i^B(p)$ is continuous on P, so $z^B(p)$ is continuous on P. Moreover, by the local non-satiation, $p \cdot \xi_i^B(p) = p \cdot \omega_i$ for all $p \in P$.. Therefore, $p \cdot z(p) = 0$ for all $p \in P$. i.e. z^B satisfies the assumption of Gale-Debreu-Nikaido lemma.

Step 3: Apply Gale-Debreu-Nikaido lemma:
there is a $\bar{p} \in P$ such that $z^B(\bar{p}) \leq 0$.

Step 4: Non-Binding of Debreu's box:
at prices $\bar{p}$, agents' consumption are given by

$$x_i = \xi_i^B(\bar{p}) = z_i^B(\bar{p}) + \omega_i$$

Note as $x_i \geq 0$, $x_i \leq \sum_{i \in I} x_i \leq z^B(\bar{p}) + \sum_{i \in I} \omega_i \leq \sum_{i \in I} \omega_i \leq (B, B, ..., B)$. i.e.

$$x_i = \xi_i^B(\bar{p}) = \xi_i(\bar{p})$$

Step 5: Market clearing:
If $z^B(\bar{p}) < 0$, we note that there are some excess endowment of some commodities. We give all these excess endowments to agent 1:

$$\tilde{x}_1 = x_1 - z^B(\bar{p}) \geq x_1$$

By Walras' law, $\bar{p} \cdot z^B(\bar{p}) = 0$. We have, $\bar{p} \cdot x_1 = \bar{p} \cdot \tilde{x}_1$. i.e. agent 1 can afford $\tilde{x}_1$. Moreover, by monotonicity, $u_1(\tilde{x}_1) \geq u_1(x_1)$. As x_1 is an optimal consumption, $u_1(\tilde{x}_1) = u_1(x_1)$. By the strict concavity of u_1, there can be at most one maximizer. Therefore, $\tilde{x}_1 = x_1$. i.e. $z^B(\bar{p}) = 0$. By Step 4, we note $0 = z^B(\bar{p}) = \sum_{i \in I} (\xi_i^B(\bar{p}) - \omega_i) = \sum_{i \in I} (\xi_i(\bar{p}) - \omega_i) = z(\bar{p})$. We finish the proof. ■