# Lecture 19: Core

Advanced Microeconomics I, ITAM

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Core is the stability concept in general equilibrium models: an allocation is a core allocation if no group of agents can jointly do better by retrade with each other within the group.

We will state two results related to the core. First, we state a stronger version of the first welfare theorem that competitive allocations are in the core, with the same assumption holds for the first welfare theorem. That is, competitive allocations are not only Pareto optimal, but also stable. This result strengthens the first welfare theorem by stating the extreme Pareto optimal cases, for instance all resources in economy are allocated to one person, will not appear in a competitive equilibrium. Second, we stated the celebrated core convergence theorem that core allocations of large economies are Walrasian without any transfers.[1] Taking these two results together, we obtain that competitive allocations coincide with core allocations in large economies.

Edgeworth, criticizing Walras, took the view that the core, rather than the set of Walrasian equilibrium, was the best description of possible allocations that the market mechanism produces. In particular, the definition of core does not impose that assumption that agents are price taker, as by Walras. In addition, if any allocation is not in the core, agents in some

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[1]There are numerous formulations of this statement. While the informal literature of this observation dated back to Edgeworth (1981), the formal literature began from Debreu-Scarf's work (1963) for finite large economies - core allocations are nearly Walrasian in large economies, and Aumann's work (1964) for continuum economies, the economy with a continuum of agents - core allocations are exactly Walrasian in continuum economies. While personally I think the statement of Aumann is conceptually cleaner, we have not developed enough technical tools to understand the statement. For this reason, I will give the statement of Debreu and Scarf. This literature is further developed by a number of economists, including Anderson, Bewley, Brown, Dierker, Grodal, Hildenbrand, Kannai, Khan and Vind.

group would find it in their interests to retrade. For this reason, Edgewroth argues that the core is a significant positive solution concept.

Note that the welfare theorems do not directly support the desirability of allowing "free market" to operate. Implicitly in the definition, the key assumption is that economic agents act as price takers. If this assumption is false, the welfare theorems will have little explaination power on the market issue that whether the market or planned economy produces more desirable outcomes. The fact that prices are used to equate demand and supply doesn't imply the result is Walrasian directly - an agnet with market power can affect the price by changing his supply to the market, thereby altering the market price and lead the economy to inefficiency.

However, the core convergence result provided a justification of the price taking assumption: when an (large) economy is at a stable state, trade occurs at a single price. [2]

# 1 Core

In an exchange economy $\mathcal{E}$, a *coalition* is a subset of agents specified by a set $S \subset I$. We say *coalition $S$ blocks an allocation* $(x_i)_{i\in I}$, or is a *blocking coalition* when the allocation we talk about is clear, if there is a redistribution of resources within this coalition such that nobody is worse off and someone is better off. Mathematically, a coalition $S$ blocks an allocation $(x_i)_{i\in S}$ if there is a tuple $(y_i)_{i\in S} \in (\mathbb{R}_+^N)^S$ such that

$$\sum_{i\in S} y_i = \sum_{i\in S} \omega_i$$

$$u_i(y_i) \geq u_i(x_i), \forall i \in S$$

$$u_j(y_j) > u_j(x_j), \text{for some } j \in S$$

We say an allocation is a *core allocation* if it is feasible and no coalition blocks it. In addition, we define *Core* is the set of core allocations. [3]

[2] A significant amount of this introduction is borrowed from Robert Anderson's lecture notes.

[3] As a comparison to the marriage market, a market contains a equal number of men and women, a stable (marriage) matching is a partition of the market into groups containing a man and a woman such that no blocking coalition of size 2 exists.

Taking a coalition to be the grand coalition $I$, we note as any core allocation is not blocked by coalition $I$, it must be Pareto optimal. In a two agents economy with agent A and agent B, there are three non-trivial coalitions - $\{A\}, \{B\}, \{A, B\}$. Therefore, the core of this economy is the set of Pareto optimal allocations that are individual rational.

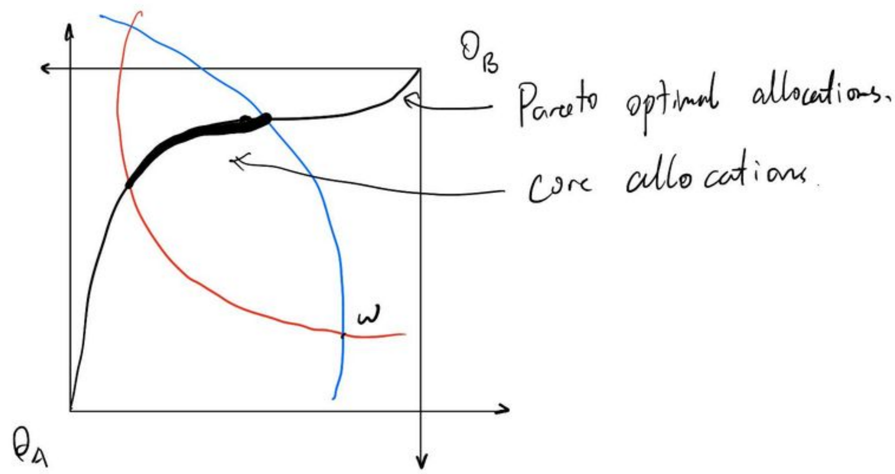


Figure 1: Core Allocations

## 2 Walrasian allocation is in the core

We first state and prove that the Walrasian allocations are stable.

**Theorem 1** (Strong First Welfare Theorem)**.** *In an exchange economy $\mathcal{E}$, if all utility functions are locally non-satiated, every Walrasian allocation is in the core.*

*Proof.* Let $(\bar{p}, (\bar{x}_i)_{i\in I})$ be a Walrasian equilibrium. If $(\bar{x}_i)_{i\in I}$ is not in the core, then there is a blocking coalition $S \subset I$ and a redistribution $(y_i)_{i\in S}$ such that

$$\sum_{i\in S} y_i = \sum_{i\in S} \omega_i \qquad (*)$$

$$u_i(y_i) \geq u_i(\bar{x}_i), \forall i \in S$$

$$u_j(y_j) > u_j(\bar{x}_j), \text{for some } j \in S$$

By the same argument as in the first welfare theorem, the two above inequalities imply,

$$\bar{p} \cdot y_i \geq \bar{p} \cdot \bar{x}_i, \forall i \in I$$

$$\bar{p} \cdot y_j > \bar{p} \cdot \bar{x}_j, \text{for some } j \in S$$

Sum them up, we have

$$\bar{p} \cdot \sum_{i \in S} y_i > \bar{p} \cdot \sum_{i \in S} \bar{x}_i = \bar{p} \cdot \sum_{i \in S} \omega_i$$

which contradicts equation (*). ■

# 3 Core Convergence

The Edgeworth's conjecture stated that the as the number of agents goes to infinity, any stable state of an economy is Walrasian.

Debreu and Scarf (1963) proved that for an economy, if we replicate the economy $r$ times, as $r \to \infty$, the core allocation of the replicated economy is nearly Walrasian. We will just give a formulation of this observation. For the proof, please check their paper.

First, an *r-replicated economy* is represented by a tuple $\mathcal{E}^r = (I, r, (u_i, \omega_i)_{i \in I})$, where $I$ is a type set of agents, each type $i$ agent is described by his utility function $u_i$ and initial endowment $\omega_i$, and there are $r$ agents of type $i$ in the replicated economy for every $i$.

We start with two observations of the replicated economy. The first observation is that at a stable state of the economy, the same type of agents consume the same bundle.

**Proposition** (Equal Treatment Property)**.** *When utility functions are strictly concave, a core allocation assigns the same consumption to all agents of the same type.*

*Proof.* For a core allocation in the $r$-replicated economy, denote the consumption of $r$ type $i$ agents by $x_i^q$, for $q = 1, 2, ..., r$. Let $x_i$ denote the least preferred consumption for type $i$ agents among $x_i^1, ..., x_i^r$. Suppose that for some type $i'$, two agents of type $i'$ have different consumption, then by the strict convexity of preferences, we have

$$\frac{1}{r} \sum_{q=1}^{r} x_i^q \succeq x_i$$

with strict inequality for type $i'$ by the strict convexity of preference. By the market clearing condition,

$$\sum_{i \in I} (\frac{1}{r} \sum_{q=1}^{r} x_i^q) = \sum_{i \in I} \omega_i$$

Therefore, a coalition containing one agent of each type, each of whom receives a least preferred consumption, would block. ■

The second observation is, as the economy is replicated, the core will shrink.

**Proposition.** *The core of $\mathcal{E}^r$ is a subset of the core of $\mathcal{E}^{r+1}$.*

*Proof.* Any blocking coalition in $\mathcal{E}^r$ is a blocking coalition in $\mathcal{E}^{r+1}$. ■

The first observation allows us to represent the core allocation in any replicated economy, we just need to specify a consumption for each type of agent, instead of every agent. i.e. A core allocation in any $r$-replicated economy can be represented by $(x_1, ..., x_I)$. The second observation states that a core allocation of a large replicated economy can be described by an allocation $(x_1, ..., x_I)$ such that it is in the core of all replicated economy.

Now, we are ready to state the core convergence theorem:

**Theorem 2** (Debreu and Scarf, 1963)**.** *For an economy $\mathcal{E}$ where all utility functions are locally non-satiated, strictly concave, continuous and the initial endowments are strict positive, if $(x_1, ..., x_I)$ is a core allocation for all replicated economies, then it is a competitive allocation.*