Lecture 4: Real Analysis

Advanced Microeconomics I, ITAM

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In this section, we give a few basic concepts in topology. These concepts will appear in calculus and optimizations.

1 Metric on $\mathbb{R}^n$

Given a linear space X, a *norm* on X is a function $p : X \to \mathbb{R}$ satisfies the following conditions:

- $p(cx) = |c|p(x)$, for all $x \in X$, $c \in \mathbb{R}$
- $p(x) + p(y) \geq p(x + y)$, for all $x, y \in X$
- $p(x) = 0$ implies $x = 0$

When $X = \mathbb{R}^n$, we have the following natural norm: for any vector $x \in \mathbb{R}^n$, we define its norm by

$$\|x\| = \sqrt{x_1^2 + ... + x_n^2}$$

It is an exercise to verify $\|\cdot\|$ is a norm. Moreover, in $\mathbb{R}^n$, the distance between two points x and y are given by

$$\|x - y\| = \sqrt{\|x_1 - y_1\|^2 + ... + \|x_n - y_n\|^2}$$

Remark. *It is easy to verify that the 1-norm defined by*

$$\|x\|_1 = |x_1| + ... + |x_n|$$

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is also a norm on $\mathbb{R}^n$. And the norm we defined above is sometimes called a 2-norm. There are reasons that a 2-norm is preferred to an 1-norm. For instance, $\|x\|_1$ is not a differentiable function on $\mathbb{R}^n$ at the origin, while $\|x\|^2$ is differentiable everywhere, and $\nabla_x \|x\|^2 = 2x$. For similar reasons, the unit ball with respect to the 1-norm will have kinks on its boundary, while the unit ball with respect to the 2-norm is has a smooth boundary.

2 Open and Closed

For any $\varepsilon > 0$, we define the *open ball of radius ε about y* in $\mathbb{R}^n$ to be

$$B_\varepsilon(y) = \{x : \|x - y\| < \varepsilon\}$$

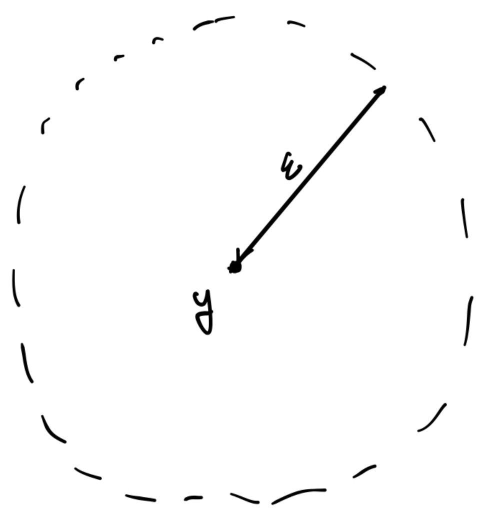

Figure 1: open ball of radius ε about y

For any $S \subset \mathbb{R}^n$, we say S is *open* if for all $x \in S$, there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset S$. i.e.every point in an open set is surrounded by some open ball.

Example 1. *It is an exercise to verify the following examples.*

- $\mathbb{R}^n, \emptyset$ *are open*
- $(0, 1)$ *is open*

- $[0, 1]$ *is not open*

- $\{(x, 0) : x \in (0, 1)\}$ *is not open*

- $B_\varepsilon(y)$ *is open*

Next we define closed set: for any $S \subset \mathbb{R}^n$, we say S is *closed* if $S^c = \mathbb{R}^n \backslash S$ is open.

Example 2. *It is an exercise to verify the following examples.*

- $\mathbb{R}^n, \emptyset$ *are closed*

- $(0, 1)$ *is not closed*

- $[0, 1]$ *is closed*

- $\{(x, 0) : x \in [0, 1]\}$ *is closed*

3 Limit

Recall, given a nonempty set $X \subset \mathbb{R}^n$, a *sequence* in set X is a function from $\mathbb{N}$ to X, which is usually denoted by x_n, or $x_1, ..., x_n, ...$

For instance, $x_n = \sin(n^2)$ is a sequence in $\mathbb{R}$.

We say a sequence x_n *converges to* x in X if for any $\varepsilon > 0$, there is a natural number N such that for any $n > N$,

$$\|x_n - x\| < \varepsilon$$

When the sequence x_n converges to x, we say the sequence x_n is *convergent*, and the *limit* of x_n is x, written as $x_n \to x$, or $\lim_{n\to+\infty} x_n = x$.

Example 3.

- $\lim_{n\to+\infty} \frac{1}{n} = 0$

- $\lim_{n\to+\infty} \frac{n-1}{n+1} = 1$

- $x_n = \sin n^2$ *does not have a limit.*

Remark. *Sometimes, on $\mathbb{R}$, we say a sequence tends to infinity or negative infinity, written as $x_n \to +\infty$ or $x_n \to -\infty$. For the first case, it means for all big number $M > 0$, we can find a natural number N such that for all $n > N$, $x_n > M$. Similarly, for the second case, it means for all negatively large number $M < 0$, we can find a natural number N such that for all $n > N$, $x_n < M$.*

Proposition. *Given a nonempty set $S \subset \mathbb{R}^n$, S is closed if and only if all convergent sequences in S converges to a point in S.*

Proof. Exercise. ■

Given a non-empty set $S \subset \mathbb{R}^n$, we define

- Interior: the interior of S is defined by

$$int(S) = \mathring{S} = \{x \in S : x \text{ is surrounded by an open ball in } S\}$$

- closure: the closure of S is defined by

$$cl(S) = \bar{S} = \{x \in \mathbb{R}^n : x_n \to x, \text{ for some sequence } x_n \text{ in } S\}$$

- boundary: the boundary of S is defined by

$$\partial S = cl(S) - int(S) = \{x \in \mathbb{R}^n : x \in cl(S), x \notin int(S)\}$$

Example 4.

- $S \subset cl(S)$
- *$int(S)$ is open, $cl(S), \partial S$ are closed*
- $int(\{x\}) = \emptyset$
- $\partial B_1(0) = \{x \in \mathbb{R}^n : \|x\| = 1\}$
- $int[a, b] = (a, b)$, $cl(a, b) = [a, b]$, $\partial(a, b) = a, b$
- *When $S = (a, b] \cup c$, where $c > b > a$, $int(S) = (a, b)$, $cl(S) = [a, b] \cup c$, $\partial S = \{a, b, c\}$.*

Exercise. *Verify the above examples and prove the above properties.*

4 Compactness

Given a non-empty set $S \subset \mathbb{R}^n$, we say S is *bounded* if there is an $M > 0$ such that $\|x\| \leq M$ for all $x \in S$. Moreover, we say S is *compact* if it is closed and bounded.

c.f. A sequence is *bounded* if $\|x_n\| \leq M$ for some $M > 0$.

Bounded sets and compact sets possess good properties. To describe these property, we first define the subsequence of a sequence: A *subsequence* of a sequence x_n is a sequence in the form $x_{n_1}, x_{n_2}, ...$, where $n_{k+1} > n_k$.

Theorem. 1. *If $x_n \to x$, then every subsequence of x_n converges to x.*

2. *Every convergent sequence in $\mathbb{R}^n$ is bounded.*

3. *(Bolzano-Weierstrass) If x_n is a bounded sequence in $\mathbb{R}^n$, then it has a convergent subsequence.*

4. *If all convergent subsequences of a sequence converge to the same point, this sequence also converges to the same point.*

5. *In a compact set, every sequence has a convergent subsequence that converges to a point in the set.*

A relevant example for this theorem is $x_n = (-1)^n$. We note, $x_1, x_3, x_5, ...$ is a convergent subsequence of x_n converges to -1, and $x_2, x_4, x_6, ...$ is a convergent subsequence of x_n converges to 1.

Proof. 1,2,4,5 are left as an exercise. The proof of number 3 involves repeated cutting the range of sequences into half. For the ease of writing, I only sketch the proof for sequence in $\mathbb{R}$. In $\mathbb{R}^n$ it is the same. To start with, say the bounded sequence x_n are in the interval $[-M, M]$. Then, we consider the two subintervals $[-M, 0]$ and $[0, M]$, at least one interval will have infinitely many points in $\{x_n : n \in \mathbb{N}\}$. Say such an interval is $[-M, 0]$. Then we consider its two subintervals $[-M, -M/2]$ and $[-M/2, 0]$. Again, there is at least one such subinterval containing an infinite number of points in x_n. Keep doing this process, the size of the subinterval goes to zero, but there is still some subinterval containing an infinite number of points in $\{x_n : n \in \mathbb{N}\}$. Doing this process enough times, and we have the convergent subsequence defined as the point in a subinterval that is very small. ■

5 Continuous function

5.1 Functions

Recall that given two sets X and Y, a *function from X and Y* is a map specifying one and only one value $f(x) \in Y$ for each $x \in X$, denoted by $f : X \to Y$. Here, X is called the *domain* of f, Y is called the *range* of f, and $f(X) = \{f(x) : x \in X\}$ is called the *image* of f. When $Y = \mathbb{R}$, f is called a real-valued function on X. If $f(x) = y$, we say y is an image of x under function f, and x is a preimage of y under function f.

Example 5.

- *For a real-valued function on $\mathbb{R}$ defined by $f(x) = |x|$, the image of this function is $[0, \infty)$.*
- *For a real-valued function on $\mathbb{R}$ defined by $f(x) = \sin x^2$, the image of this function is $[-1, 1]$.*

Next, we define injectivity, sujectivity and bijectivity.

- A function $f : X \to Y$ is *injective*, or *one to one*, if $f(x) = f(y)$ implies $x = y$.
  - e.g. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is not injective
  - e.g. $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = |x|$ is injective
- A function $f : X \to Y$ is *surjective*, or *onto*, if $f(X) = Y$. i.e. for all $y \in Y$, there is an $x \in X$ such that $f(x) = y$.
  - e.g. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is not surjective
  - e.g. $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = |x|$ is surjective
- A function $f : X \to Y$ is *bijective*, or *one to one and onto*, if f is both surjective and injective. i.e. for all $y \in Y$, there is a unique $x \in X$ such that $f(x) = y$.

Last, we define the increasingness and decreasingness of a function. For a real valued function f on $\mathbb{R}$, we say f is an *increasing function* in $S \subset \mathbb{R}$ if $f(x) \geq f(y)$ if and only if $x \geq y$ in S. In contrast, we say f is a *decreasing function* in $S \subset \mathbb{R}$ if $f(x) \geq f(y)$ if and only if $x \leq y$ in S. When a function is increasing (or decreasing) in $\mathbb{R}$, we say it is an increasing (or decreasing) function.

5.2 Continuity

Given nonempty set $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, $f : X \to Y$ is a continuous function if for all $x \in X$, and any convergent sequence $x_n \to x$ in X,

$$f(x_n) \to f(x)$$

Example 6.

- $f : (0, \infty) \to \mathbb{R}$, *where* $f(x) = \log x$ *is continuous*
- $f : (0, \infty) \to \mathbb{R}$, *where* $f(x) = 1/x$ *is continuous*
- $f : \mathbb{R} \to \mathbb{R}$, *where* $f(x) = 1/x$ *for* $x \neq 0$ *and* $f(0) = 0$ *is discontinuous at* 0
- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = sgn(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$ *is discontinuous at* 0.

Continuous function has optima in a compact set.

Theorem. *For a continuous function* $f : X \to \mathbb{R}$, *if* $X \subset \mathbb{R}^n$ *is compact,* f *has a maximum and a minimum in* X.

Remark. *Recall that a compact set in* $\mathbb{R}^n$ *is bounded and closed. Both the boundedness and the closedness are indispensable for the existence of a maximizer and a minimizer. For instance, when* X *is only closed but is not bounded, then the real-valued function* $f(x) = x$ *on* $\mathbb{R}$ *has no optimum. When* X *is only bounded but is not closed, then the real-valued function* $f(x) = 1/x$ *on* $(0, 1)$ *has no optimum.*

Proof. We will only prove that f has a maximum on some compact set X. The minimum part is exactly the same. The proof contains two steps.

First, we observe that $f(X)$ is compact when X is compact. To start with, $f(X)$ is closed as f is continuous and X is closed. Next, to see it is bounded, we prove by contradiction. Assuming that $f(X)$ is not bounded from above. (the case that $f(X)$ is not bounded from below is the same.) i.e. there are a sequence x_n in X such that $f(x_n) \to +\infty$. Since X is compact, x_n has a convergent subsequence converging in X. That is, $x_{n_k} \to x$ for some subsequence x_{n_k} and some $x \in X$. By continuity, $f(x_{n_k})$ must be bounded, as $f(x) \in \mathbb{R}$. Therefore, we have a contradiction.

Second, we prove that a maximizer exists. Since $f(X)$ is compact, in particular bounded, we know that $M = \sup f(X) \in \mathbb{R}$. By the definition of supremum, we have a sequence x_n in X such that $f(x_n) \to M$. By the compactness of X, x_n have a convergent subsequence $x_{n_k} \to x$ such that $f(x_{n_k}) \to M$. By continiuty, $f(x) = \sup f(X)$. i.e. $x \in X$ is a maximizer. ■

Lastly, we impose a nice observation that a real-valued convex function on an open set always exhibit some regularity.

Theorem. *When X is convex and open, a real-valued convex function on X is continuous.*

Proof. This could be an interesting exercise for interested readers. ■

Remark.

- *The conclusion does not hold if X is not open. For instance, the real-valued function f defined on $[0,1]$ by* $f = \begin{cases} x^2 & x \in (0,1] \\ -1 & x = 0 \end{cases}$ *is not continuous.*
- *(Alexandrov[1]) Indeed, a convex function on an open set is almost everywhere second order differentiable.*[2]

[1] See Theorem 14.25 in Villani's book on Optimal transport in 2009.

[2] It means, whenever you are able to (uniformly) randomly select a point in the domain of the function, the function has a second order derivative at that point with probability one.