Rationale Exponenten

Theorie

Allgemeine Wurzeln können auch als Potenzen mit rationalem Exponenten dargestellt werden.

$$\sqrt{a} = a^{\frac{1}{2}} \qquad \sqrt[n]{a} = a^{\frac{1}{n}} \qquad \sqrt[n]{a^k} = \left(\sqrt[n]{a}\right)^k = a^{\frac{k}{n}}$$

Dabei gelten die bekannten Potenzgesetze:

$$a^p \cdot a^q = a^{p+q} \qquad a^p \cdot b^p = (a \cdot b)^p \qquad (a^p)^q = a^{p \cdot q}$$

$$\frac{a^p}{a^q} = a^{p-q} \qquad \frac{a^p}{b^p} = \left(\frac{a}{b}\right)^p$$

1. **Rationale Exponenten.** Vereinfachen Sie und stellen Sie als Potenz mit rationalem Exponenten dar:

   Beispiel: $\sqrt[4]{144} = \sqrt[4]{12^2} = 12^{\frac{2}{4}} = 12^{\frac{1}{2}}$

   (a) $\sqrt[6]{625} = \sqrt[6]{5^4} = 5^{\frac{4}{6}} = 5^{\frac{2}{3}}$

   (b) $\sqrt[9]{125} = \sqrt[9]{5^3} = 5^{\frac{3}{9}} = 5^{\frac{1}{3}}$

   (c) $\sqrt[10]{32} = \sqrt[10]{2^5} = 2^{\frac{5}{10}} = 2^{\frac{1}{2}}$

   (d) $\sqrt[3]{27} = \sqrt[3]{3^3} = 3^{\frac{3}{3}} = 3$

   (e) $\sqrt[4]{81} = \sqrt[4]{3^4} = 3^{\frac{4}{4}} = 3$

   (f) $\sqrt[5]{32} = \sqrt[5]{2^5} = 2^{\frac{5}{5}} = 2$

   (g) $\sqrt[3]{64} = \sqrt[3]{8^2} = 8^{\frac{2}{3}}$

   (h) $\sqrt[6]{4} = \sqrt[6]{2^2} = 2^{\frac{2}{6}} = 2^{\frac{1}{3}}$

   (i) $\sqrt[4]{36} = \sqrt[4]{6^2} = 6^{\frac{2}{4}} = 6^{\frac{1}{2}}$

2. **Als Wurzel.** Vereinfachen Sie und stellen Sie die Zahl als Wurzel mit möglichst kleinem Wurzelexponenten dar.

   Beispiel: $8^{\frac{1}{6}} = \left(2^3\right)^{\frac{1}{6}} = 2^{3 \cdot \frac{1}{6}} = 2^{\frac{3}{6}} = 2^{\frac{1}{3}} = \sqrt[3]{2}$

   (a) $25^{\frac{1}{8}} = \left(5^2\right)^{\frac{1}{8}} = 5^{2 \cdot \frac{1}{8}} = 5^{\frac{2}{8}} = 5^{\frac{1}{4}} = \sqrt[4]{5}$

   (b) $125^{\frac{1}{6}} = \left(5^3\right)^{\frac{1}{6}} = 5^{\frac{3}{6}} = 5^{\frac{1}{2}} = \sqrt{5}$

   (c) $27^{\frac{1}{9}} = \left(3^3\right)^{\frac{1}{9}} = 3^{3 \cdot \frac{1}{9}} = 3^{\frac{3}{9}} = 3^{\frac{1}{3}} = \sqrt[3]{3}$

(d) $16^{\frac{1}{12}} = \left(2^4\right)^{\frac{1}{12}} = 2^{4\cdot\frac{1}{12}} = 2^{\frac{4}{12}} = 2^{\frac{1}{3}} = \sqrt[3]{2}$

(e) $81^{\frac{1}{10}} = \left(3^4\right)^{\frac{1}{10}} = 3^{4\cdot\frac{1}{10}} = 3^{\frac{4}{10}} = 3^{\frac{2}{5}} = \sqrt[5]{3^2} = \sqrt[5]{9}$

(f) $81^{\frac{1}{20}} = \left(3^4\right)^{\frac{1}{20}} = 3^{4\cdot\frac{1}{20}} = 3^{\frac{4}{20}} = 3^{\frac{1}{5}} = \sqrt[5]{3}$

(g) $216^{\frac{1}{6}} = \left(6^3\right)^{\frac{1}{6}} = 6^{3\cdot\frac{1}{6}} = 6^{\frac{3}{6}} = 6^{\frac{1}{2}} = \sqrt{6}$

(h) $32^{\frac{1}{15}} = \left(2^5\right)^{\frac{1}{15}} = 2^{5\cdot\frac{1}{15}} = 2^{\frac{5}{15}} = 2^{\frac{1}{3}} = \sqrt[3]{2}$

(i) $256^{\frac{1}{8}} = \left(2^8\right)^{\frac{1}{8}} = 2^{\frac{8}{8}} = 2$

3. **Berechnen** Sie ohne Taschenrechner.

(a) $\sqrt{2} \cdot \sqrt[3]{2} \cdot \sqrt[6]{2}$

Lösung:

$$
\begin{aligned}
&= 2^{\frac{1}{2}} \cdot 2^{\frac{1}{3}} \cdot 2^{\frac{1}{6}} \\
&= 2^{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}} \\
&= 2^{\frac{3}{6}+\frac{2}{6}+\frac{1}{6}} \\
&= 2^{\frac{6}{6}} \\
&= 2^1 \\
&= 2
\end{aligned}
$$

(b) $\sqrt{6} \cdot \sqrt[3]{12} \cdot \sqrt[6]{96}$

Lösung:

$$
\begin{aligned}
&= 6^{\frac{1}{2}} \cdot (2\cdot 6)^{\frac{1}{3}} \cdot \left(2^4 \cdot 6\right)^{\frac{1}{6}} \\
&= 2^{\frac{1}{3}} \cdot \left(2^4\right)^{\frac{1}{6}} \cdot 6^{\frac{1}{2}} \cdot 6^{\frac{1}{3}} \cdot 6^{\frac{1}{6}} \\
&= 2^{\frac{1}{3}+\frac{4}{6}} \cdot 6^{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}} \\
&= 2^1 \cdot 6^1 \\
&= 12
\end{aligned}
$$

(c) $\sqrt{10} \cdot \sqrt[3]{5} \cdot \sqrt[4]{2} \cdot \sqrt[12]{200}$

Lösung:

$$
\begin{aligned}
&= 10^{\frac{1}{2}} \cdot 5^{\frac{1}{3}} \cdot 2^{\frac{1}{4}} \cdot 200^{\frac{1}{12}} \\
&= (2 \cdot 5)^{\frac{1}{2}} \cdot 5^{\frac{1}{3}} \cdot 2^{\frac{1}{4}} \cdot \left(2^3 \cdot 5^2\right)^{\frac{1}{12}} \\
&= 2^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} \cdot 5^{\frac{1}{3}} \cdot 2^{\frac{1}{4}} \cdot \left(2^3\right)^{\frac{1}{12}} \cdot \left(5^2\right)^{\frac{1}{12}} \\
&= 2^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} \cdot 5^{\frac{1}{3}} \cdot 2^{\frac{1}{4}} \cdot 2^{\frac{3}{12}} \cdot 5^{\frac{2}{12}} \\
&= 2^{\frac{1}{2}+\frac{1}{4}+\frac{3}{12}} \cdot 5^{\frac{1}{2}+\frac{1}{3}+\frac{2}{12}} \\
&= 2^{\frac{2}{4}+\frac{1}{4}+\frac{1}{4}} \cdot 5^{\frac{3}{6}+\frac{2}{6}+\frac{1}{6}} \\
&= 2^{\frac{4}{4}} \cdot 5^{\frac{6}{6}} \\
&= 2^1 \cdot 5^1 \\
&= 10
\end{aligned}
$$

(d) $\sqrt{2} \cdot \sqrt[6]{2} - \sqrt[3]{4}$

Lösung:

$$
\begin{aligned}
&= 2^{\frac{1}{2}} \cdot 2^{\frac{1}{6}} - 4^{\frac{1}{3}} \\
&= 2^{\frac{1}{2}+\frac{1}{6}} - \left(2^2\right)^{\frac{1}{3}} \\
&= 2^{\frac{3}{6}+\frac{1}{6}} - 2^{2 \cdot \frac{1}{3}} \\
&= 2^{\frac{4}{6}} - 2^{\frac{2}{3}} \\
&= 2^{\frac{2}{3}} - 2^{\frac{2}{3}} \\
&= 0
\end{aligned}
$$

(e) $\left(\sqrt[3]{54} - \sqrt[3]{16}\right) \cdot \sqrt[6]{16}$

Lösung:

$$
\begin{aligned}
&= \left(54^{\frac{1}{3}} - 16^{\frac{1}{3}}\right) \cdot 16^{\frac{1}{6}} \\
&= \left(\left(2 \cdot 3^3\right)^{\frac{1}{3}} - \left(2^4\right)^{\frac{1}{3}}\right) \cdot \left(2^4\right)^{\frac{1}{6}} \\
&= \left(2^{\frac{1}{3}} \cdot 3^{\frac{3}{3}} - 2^{\frac{4}{3}}\right) \cdot 2^{\frac{4}{6}} \\
&= \left(2^{\frac{1}{3}} \cdot 3 - 2^{\frac{4}{3}}\right) \cdot 2^{\frac{2}{3}} \\
&= 2^{\frac{1}{3}} \cdot 3 \cdot 2^{\frac{2}{3}} - 2^{\frac{4}{3}} \cdot 2^{\frac{2}{3}} \\
&= 2^{\frac{1}{3}+\frac{2}{3}} \cdot 3 - 2^{\frac{4}{3}+\frac{2}{3}} \\
&= 2^{\frac{3}{3}} \cdot 3 - 2^{\frac{6}{3}} \\
&= 2^1 \cdot 3 - 2^2 \\
&= 6 - 4 \\
&= 2
\end{aligned}
$$

(f) $\left(\sqrt[4]{512} - \sqrt[4]{32} - \sqrt[4]{2}\right) \cdot \sqrt[4]{8}$

Lösung:

$$
\begin{aligned}
&= \left(512^{\frac{1}{4}} - 32^{\frac{1}{4}} - 2^{\frac{1}{4}}\right) \cdot 8^{\frac{1}{4}} \\
&= \left(\left(2^9\right)^{\frac{1}{4}} - \left(2^5\right)^{\frac{1}{4}} - 2^{\frac{1}{4}}\right) \cdot \left(2^3\right)^{\frac{1}{4}} \\
&= \left(2^{\frac{9}{4}} - 2^{\frac{5}{4}} - 2^{\frac{1}{4}}\right) \cdot 2^{\frac{3}{4}} \\
&= 2^{\frac{9}{4}} \cdot 2^{\frac{3}{4}} - 2^{\frac{5}{4}} \cdot 2^{\frac{3}{4}} - 2^{\frac{1}{4}} \cdot 2^{\frac{3}{4}} \\
&= 2^{\frac{9}{4}+\frac{3}{4}} - 2^{\frac{5}{4}+\frac{3}{4}} - 2^{\frac{1}{4}+\frac{3}{4}} \\
&= 2^{\frac{12}{4}} - 2^{\frac{8}{4}} - 2^{\frac{4}{4}} \\
&= 2^3 - 2^2 - 2^1 \\
&= 8 - 4 - 2 \\
&= 2
\end{aligned}
$$

4. Vereinfachen Sie und stellen Sie das Resultat sowohl in Potenz- als auch in Wurzelschreibweise dar.

(a) $3^{0.5} \cdot 9^{0.75}$

Lösung:

$$= 3^{\frac{1}{2}} \cdot \left(3^2\right)^{\frac{3}{4}}$$
$$= 3^{\frac{1}{2}} \cdot 3^{\frac{6}{4}}$$
$$= 3^{\frac{1}{2}+\frac{6}{4}}$$
$$= 3^{\frac{1}{2}+\frac{3}{2}}$$
$$= 3^2$$
$$= 9$$

(b) $4^{\frac{1}{5}} \cdot 2^{\frac{1}{10}} : 8^{\frac{1}{4}}$

Lösung:

$$= \left(2^2\right)^{\frac{1}{5}} \cdot 2^{\frac{1}{10}} : \left(2^3\right)^{\frac{1}{4}}$$
$$= 2^{\frac{2}{5}} \cdot 2^{\frac{1}{10}} : 2^{\frac{3}{4}}$$
$$= 2^{\frac{2}{5}+\frac{1}{10}-\frac{3}{4}}$$
$$= 2^{\frac{8}{20}+\frac{2}{20}-\frac{15}{20}}$$
$$= 2^{-\frac{5}{20}}$$
$$= 2^{-\frac{1}{4}} = \frac{1}{\sqrt[4]{2}}$$

(c) $\sqrt[4]{27} \cdot \sqrt[5]{9} \cdot \sqrt[20]{27}$

Lösung:

$$= 27^{\frac{1}{4}} \cdot 9^{\frac{1}{5}} \cdot 27^{\frac{1}{20}}$$
$$= \left(3^3\right)^{\frac{1}{4}} \cdot \left(3^2\right)^{\frac{1}{5}} \cdot \left(3^3\right)^{\frac{1}{20}}$$
$$= 3^{\frac{3}{4}} \cdot 3^{\frac{2}{5}} \cdot 3^{\frac{3}{20}}$$
$$= 3^{\frac{3}{4}+\frac{2}{5}+\frac{3}{20}}$$
$$= 3^{\frac{15}{20}+\frac{8}{20}+\frac{3}{20}}$$
$$= 3^{\frac{26}{20}}$$
$$= 3^{\frac{13}{10}} = \sqrt[10]{3^{13}}$$

(d) $\sqrt[3]{4} : \sqrt[3]{36}$

Lösung:

$$= 4^{\frac{1}{3}} : 36^{\frac{1}{3}}$$
$$= \left(2^2\right)^{\frac{1}{3}} : \left(6^2\right)^{\frac{1}{3}}$$
$$= 2^{\frac{2}{3}} : 6^{\frac{2}{3}}$$
$$= (2 : 6)^{\frac{2}{3}}$$
$$= \left(\tfrac{1}{3}\right)^{\frac{2}{3}} = \frac{1}{3^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{3^2}}$$

(e) $\sqrt[3]{\frac{4}{5}} : \sqrt[6]{25}$

Lösung:

$$= \left(\tfrac{4}{5}\right)^{\frac{1}{3}} : 25^{\frac{1}{6}}$$
$$= \left(\tfrac{4}{5}\right)^{\frac{1}{3}} : \left(5^2\right)^{\frac{1}{6}}$$
$$= \left(\tfrac{4}{5}\right)^{\frac{1}{3}} : 5^{\frac{2}{6}}$$
$$= \left(\tfrac{4}{5}\right)^{\frac{1}{3}} : 5^{\frac{1}{3}}$$
$$= \left(\tfrac{4}{5} : 5\right)^{\frac{1}{3}}$$
$$= \left(\tfrac{4}{25}\right)^{\frac{1}{3}}$$
$$= \left(\frac{2^2}{5^2}\right)^{\frac{1}{3}}$$
$$= \left(\left(\tfrac{2}{5}\right)^2\right)^{\frac{1}{3}}$$
$$= \left(\tfrac{2}{5}\right)^{\frac{2}{3}}$$

$$= \frac{20^{\frac{1}{3}}}{5} = \frac{\sqrt[3]{20}}{5}$$

(f) $\sqrt[5]{\frac{8}{9}} : \sqrt[5]{\frac{27}{128}} = \frac{4}{3}$

5. Vereinfachen Sie so weit wie möglich und stellen Sie in Potenz- und Wurzelschreibweise dar.

Beispiel: $\left(\sqrt[4]{16}\right)^3 = 16^{\frac{3}{4}} = \left(2^4\right)^{\frac{3}{4}} = 2^{4\cdot\frac{3}{4}} = 2^3 = 8$

(a) $\sqrt[4]{3^{16}} = 3^4 = 81$

(b) $\sqrt{\sqrt[3]{2}} = 2^{\frac{1}{2}\cdot\frac{1}{3}} = 2^{\frac{1}{6}} = \sqrt[6]{2}$

(c) $\sqrt[3]{\sqrt[4]{2}} = 2^{\frac{1}{3}\cdot\frac{1}{4}} = 2^{\frac{1}{12}} = \sqrt[12]{2}$

(d) $\sqrt{\sqrt[3]{32}} = 2^{\frac{5}{6}} = \sqrt[6]{2^5}$

(e) $\sqrt[3]{\sqrt[5]{\sqrt[4]{16}}} = 2^{\frac{1}{15}} = \sqrt[15]{2}$

(f) $\sqrt{\sqrt{625}} = \left(625^{\frac{1}{2}}\right)^{\frac{1}{2}} = 625^{\frac{1}{4}} = (5^4)^{\frac{1}{4}} = 5^{\frac{4}{4}} = 5$

(g) $\sqrt[3]{4 \cdot \sqrt[4]{8}} = 2^{\frac{11}{12}} = \sqrt[12]{2^{11}}$

(h) $\sqrt[5]{9 \cdot \sqrt[3]{3 \cdot \sqrt{27}}} = 3^{\frac{17}{30}} = \sqrt[30]{3^{17}}$

(i) $\sqrt[6]{6 \cdot \sqrt[4]{6 \cdot \sqrt[3]{6}}} = 6^{\frac{2}{9}} = \sqrt[9]{6^2}$

6. Wurzelterme. Vereinfachen Sie.

(a) $a^2 \cdot \sqrt{a} = a^2 \cdot a^{\frac{1}{2}} = a^{2+\frac{1}{2}} = a^{\frac{5}{2}}$

(b) $c \cdot \sqrt[3]{c} = \sqrt[3]{c^3} \cdot \sqrt[3]{c} = \sqrt[3]{c^3 \cdot c} = \sqrt[3]{c^4}$

(c) $\sqrt{k} \cdot \sqrt[3]{k} = \sqrt{\sqrt[3]{k^3}} \cdot \sqrt[3]{\sqrt{k^2}} = \sqrt[6]{k^3} \cdot \sqrt[6]{k^2} = \sqrt[6]{k^3 \cdot k^2} = \sqrt[6]{k^5}$

(d) $\sqrt[5]{s^3} \cdot \sqrt[5]{s^2} = \sqrt[5]{s^3 \cdot s^2} = \sqrt[5]{s^5} = s$

(e) $\frac{e}{\sqrt{e}} = \frac{\sqrt{e^2}}{\sqrt{e}} = \sqrt{\frac{e^2}{e}} = \sqrt{e}$

(f) $\frac{b^2}{\sqrt[4]{b}} = \frac{\sqrt[4]{b^8}}{\sqrt[4]{b}} = \sqrt[4]{\frac{b^8}{b}} = \sqrt[4]{b^7}$

(g) $\frac{\sqrt[3]{d}}{\sqrt[4]{d}} = \frac{\sqrt[3]{\sqrt[4]{d^4}}}{\sqrt[4]{\sqrt[4]{d^3}}} = \frac{\sqrt[12]{d^4}}{\sqrt[12]{d^3}} = \sqrt[12]{\frac{d^4}{d^3}} = \sqrt[12]{d}$

(h) $\frac{\sqrt[3]{y^2}}{y} = \frac{\sqrt[3]{y^2}}{\sqrt[3]{y^3}} = \sqrt[3]{\frac{y^2}{y^3}} = \sqrt[3]{\frac{1}{y}} = \frac{1}{\sqrt[3]{y}}$

(i) $\left(\sqrt[4]{a}\right)^2 = \left(\sqrt{\sqrt{a}}\right)^2 = \sqrt{a}$

(j) $\left(\sqrt[3]{x}\right)^9 = x^3$

(k) $\sqrt[3]{\sqrt{u}} = \sqrt[6]{u}$

(l) $\sqrt{\sqrt[5]{z^4}} = \sqrt[5]{z^2}$