Geometry Behind Why Logarithms Appear in Trigonometric Integrals

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Abstract

In introductory calculus, the integral of the tangent function, $\int \tan(x)\,dx = \ln|\sec(x)|$, is standardly proven via algebraic substitution. However, this algebraic manipulation obscures the underlying geometric reality. This paper presents a purely geometric construction that demonstrates why Euler's number e and the natural logarithm naturally emerge from this integral. By discretizing the angle x and modeling the secant length as a process of continuous, non-linear multiplicative growth, we show that the natural logarithm is the necessary inverse operation of continuous geometric compounding.

1 Introduction

When evaluating $\int \tan(x)\,dx$, the standard procedure dictates rewriting the integrand as $\frac{\sin(x)}{\cos(x)}$ and applying the substitution $u = \cos(x)$. While rigorous, this method fails to intuitively explain *why* a transcendental logarithmic function describes the area under a trigonometric curve.

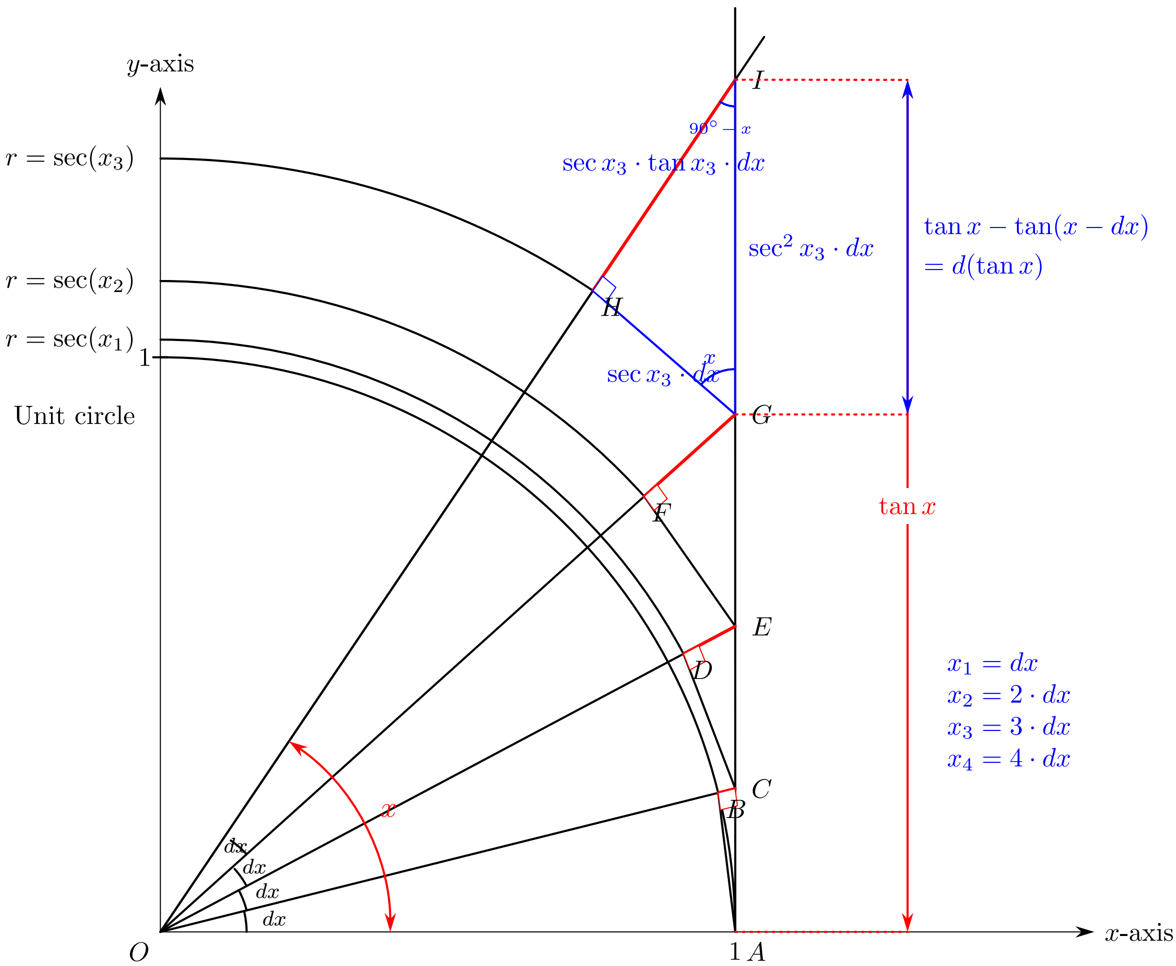
To understand this deeply, we must shift our perspective from calculating areas to tracking geometric growth. By constructing the secant line incrementally, we will reveal that $\sec(x)$ grows via continuous compounding, bridging the gap between trigonometry and the natural logarithm.

2 Geometric Construction of the Secant Function

Consider a full cartesian plane. We wish to find the length of the secant line for an angle x, which we know evaluates to $\sec(x)$. Instead of analyzing the angle x in its entirety, we discretize it into n infinitesimally small angles dx.

$$x = n \cdot dx$$

As $n \to \infty$, dx becomes infinitesimally small. For the sake of visualization, we construct the diagram using a finite n (e.g., $n = 4$). We establish a recursive construction of circles and line intersections to map out the secant length.



As $n \to \infty$ (and thus $dx \to 0$), the sector approximations ($\triangle OAB, \triangle OCD, \triangle OEF, \triangle OGH$) become infinitesimally thin isosceles triangles. The sum of their two base angles approaches $180°$, and the remaining central angle is equal to dx. The base (chord) of these infinitesimals elegantly approaches the arc length $r \cdot dx$.

3 Algebraic Formulation

We wish to analytically trace how the length $\sec x$ is generated from scratch (how 1 compounding step by step becomes $\sec x$).

> **Why is $OI = \sec x$?**
> Because in the overall right triangle constructed at the vertical tangent $x = 1$, we have $OI \cos x = 1 \implies OI = \frac{1}{\cos x} = \sec x$.

By projecting the radii across sequential steps, we accumulate radial increments (the red segments BC, DE, FG, HI):

$$1 \cdot (OA = OB) + BC = OC = OD$$
$$OE = OD + DE = OC + DE$$
$$OG = OF + FG = OE + FG$$
$$OI = OH + HI = OG + HI$$

Substituting backwards, we trace the total length multiplicatively:

$$\begin{aligned}\sec(x_4) = OI &= HI + OH \\ &= HI + \underbrace{OF + FG}_{OH} \\ &= HI + FG + \underbrace{OD + DE}_{OF} \\ &= HI + FG + DE + \underbrace{1 + BC}_{OD}\end{aligned}$$

In the infinitesimal limit, the radial increments are given by the projection $dr_i = \sec(x_i) \cdot \tan(x_i) \cdot dx$. Using this, we evaluate the sequential sum by grouping terms:

$$\begin{aligned}\sec(x_4) &= \sec(x_3) + \sec(x_3)\tan(x_3)dx \\ &= \sec(x_3)\big(1 + \tan(x_3)dx\big) \\ &= \Big[\sec(x_2) + \sec(x_2)\tan(x_2)dx\Big]\big(1 + \tan(x_3)dx\big) \\ &= \sec(x_2)\big(1 + \tan(x_2)dx\big)\big(1 + \tan(x_3)dx\big) \\ &\vdots \\ &= \sec(x_0)\big(1 + \tan(x_0)dx\big)\big(1 + \tan(x_1)dx\big)\big(1 + \tan(x_2)dx\big)\big(1 + \tan(x_3)dx\big)\end{aligned}$$

Given $\sec(x_0) = \sec(0) = 1$, the length is expressed as the continuous product of n factors:

$$\sec(x_n) = \prod_{i=0}^{n-1} \big(1 + \tan(x_i)dx\big)$$

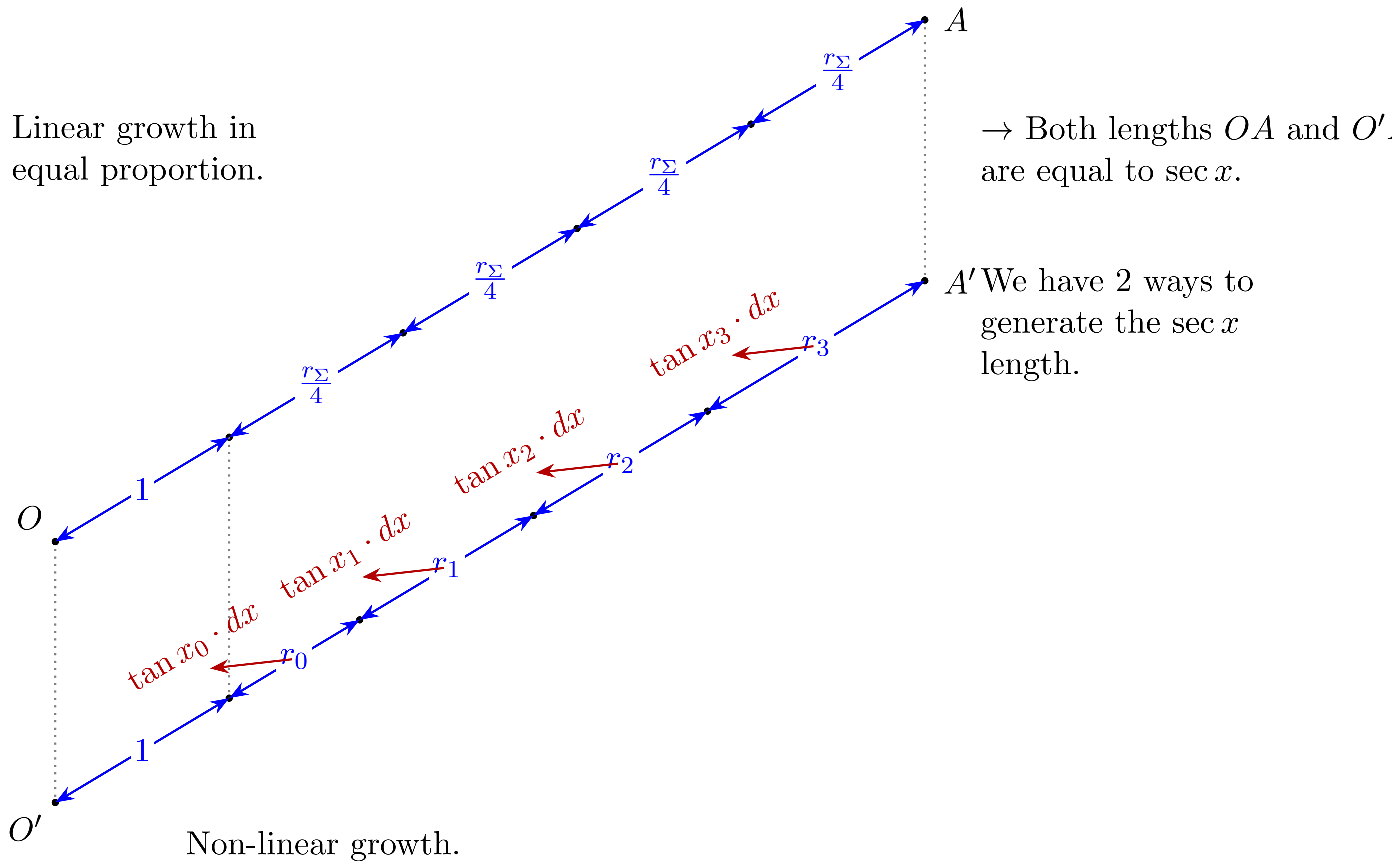
This beautifully maps out **non-linear multiplicative growth**.

4 Linearization and the Continuous Limit

Observe that there is a non-linear gap between the circles as the radius $r = \sec(x_i)$ expands. How do we linearize this?

Both paths yield a total final length equal to $\sec x$. We have two ways to conceptualize generating the $\sec x$ length. By forcing **linear growth with equal proportions**, we define the total additive rate of growth r_Σ:

$$r_\Sigma = \sum_{i=0}^{n-1} \tan(x_i)dx$$



If we replace the non-linear gaps with average, evenly divided proportions (linearizing the compounding effect), the multiplicative product is generalized and rewritten over n terms as:

$$\sec(x_n) = \prod_{i=0}^{n-1} \left(1 + \frac{\sum_{i=0}^{n-1} \tan(x_i)dx}{n}\right) = \left(1 + \frac{\sum_{i=0}^{n-1} \tan(x_i)dx}{n}\right)^n$$

As $n \to \infty$, we take the infinitesimally smallest elements, and the summation transitions to the continuous integral ($\Sigma \to \int$):

$$\sec x = \lim_{n\to\infty} \left(1 + \frac{\int_0^x \tan(t)\, dt}{n}\right)^n$$

Using the definition of the natural exponential function, $\lim_{n\to\infty} \left(1 + \frac{A}{n}\right)^n = e^A$, we obtain:

$$\sec x = e^{\int_0^x \tan t\, dt}$$

Taking the natural logarithm of both sides directly yields the fundamental result:

$$\therefore \int_0^x \tan t\, dt \;=\; \ln(\sec x)$$

5 Conclusion

The appearance of the natural logarithm in the integral of $\tan(x)$ is not merely an artifact of algebraic substitution. It is a direct requirement of the geometry. By tracing the secant line as an accumulation of rotational increments, we observe that the geometry enforces continuous multiplicative compounding, inevitably linking trigonometric proportions to Euler's number and the natural logarithm.