Lyapunov theory

The system $\dot{x} = f(x)$ is stable in the sense of Lyapunov if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that $\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \forall t \geq 0$.

The system is locally asymptotically stable (LAS) if it is stable in the sense of Lyapunov and if there exists a scalar $\delta > 0$ such that $\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$.

The system globally asymptotically stable (GAS) if it is stable in the sense of Lyapunov and $\lim_{t \to \infty} x(t) = 0 \forall x(0)$.

It is well known that the system is LAS if there exists a continuously differentiable (Lyapunov) function $V : \mathbb{R}^n \to \mathbb{R}$ that vanishes at the origin and satisfies $V(x) > 0$ and $-\langle \nabla V(x), f(x) \rangle > 0$ for all $x \in S \setminus \{0\}$, where $S$ is a neighborhood of the origin.

Moreover, if $V$ is in addition radially unbounded (i.e., satisfies $V(x) \to \infty$ when $\|x\| \to \infty$) and if $S = \mathbb{R}^n$, then the origin is GAS.

How to find such a function $V$?

- When are algebraic techniques helpful?
- What are some of their limitations?

SOS techniques to the rescue

**Idea:** Replace positivity constraint with SOS constraint. A polynomial $p(x_1, \ldots, x_n)$ is SOS (sum of squares) if it can be decomposed as

$$ p(x) = \sum_{i=1}^{m} q_i(x), \quad \text{where } q_i \text{ are polynomials.} $$

**SOS decompositions are algorithmically tractable.**

$V(x)$ and $-\nabla V(x)$ are SOS (sum of squares) if $-\nabla V(x)$ is positive semidefinite.

**Example:**

$$ \dot{x} = -\begin{pmatrix} -x^3 - xy^2 \\ -x - xy \\ -2 \end{pmatrix}, \quad \dot{y} = -\begin{pmatrix} -x - y^2 \\ -y + x^2 \\ -4 \end{pmatrix}. $$

Couple of lines of code in SOSTOOLS, YALMIP...

V = 4.1068x^2 + 5.5489y^2 + 1.7945z^2.

**Limitations**

- Polynomial Lyapunov functions may not exist ...

- $\dot{x} = -x + xy$ is GAS, but has no polynomial Lyapunov function of any degree [Ahmadi, Kirste, Parrilo].

- or they can exist but be hard to find

V(x) = $x_1^2 + x_2^2$ is a Lyapunov function, but SOS fails to find any degree 2 Lyapunov function.

A complexity theoretic limitation

Testing asymptotic stability of cubic vector fields is strongly NP-hard [Ahmadi]. Unless P=NP, small-sized algebraic certificates of stability do not exist for some stable systems.

Theorem 1: A more severe limitation

The polynomial dynamical system

$$ \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -2y(-x^4 + 2x^2y^2 + y^4) \\ 2y(x^4 + 2x^2y^2 - y^4) \end{pmatrix} \begin{pmatrix} 2x(x^4 + 2x^2y^2 - y^4) \\ 2y(-x^4 + 2x^2y^2 + y^4) \end{pmatrix} $$

is GAS but does not admit an analytic Lyapunov function even locally.

Proof idea

Consider the rational function $W(x) = \frac{1}{x^2 + y^2 - (x^2 + y^2)^2}$. The function $f$ can be written as $f = R\nabla W - (x^2 + y^2)\nabla W$, where $R$ is the rotation matrix by $\frac{\pi}{2}$ radians.

This proves that the system is GAS, as $W$ is a corresponding Lyapunov function.

Near the origin, the dominating term is $R\nabla W$, so the trajectories are close to the level sets of $W$. Therefore, any analytic Lyapunov function has to be constant on these level sets, which is impossible.

Theorem 2: A converse result for homogeneous systems

A homogeneous dynamical system $\dot{x} = f(x)$ is asymptotically stable iff it admits a rational Lyapunov function of the form

$$ V(x) = \frac{p(x)}{(\sum_{i=1}^{m} x_i^2)^r}, $$

where $p$ is a polynomial.

Furthermore, when $f$ is polynomial, $V$ can be found by SOS programming.

Proof idea

- A stable homogeneous system $f$ has a homogeneous Lyapunov functions $V$ [Rosier].
- Use Bernstein polynomials to approximate $V$ by a polynomial $p$ s.t. $p \equiv V$ and $\nabla p \approx \nabla V$ on the unit cube.
- Homogenize $p$ as $R \overset{p}{=} = \frac{1}{(\sum_{i=1}^{m} x_i^2)^r}$ for the approximation to hold on $\mathbb{R}^n$.

The degree of the rational function can be arbitrarily high

$$ \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} -2\pi y(x^2 + y^2) - 2y(2x^2 + y^4) \\ 2\pi x(x^2 + y^2) + 2x(2x^2 + y^4) \end{pmatrix} $$

GAS for $\theta \in (0, \pi)$, so a rational Lyapunov function $V_{\theta}$ exists. The degree of the numerator however has to diverge; otherwise, the limit $V_{\theta} = \lim_{\theta \to 0} V_{\theta}$ would be constant on the trajectories for $\theta = 0$, but in this case the trajectories are non-algebraic periodic orbits.