Power and Limitations of Algebraic Proofs of Stability Based on Semidefinite Programming

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Motivation

Dynamical systems are everywhere:

Walking robot Self driving cars Moons orbiting a planet

In many applications we are interested in certifying a qualitative property of a family of trajectories of a nonlinear dynamical system: stability, safety, periodicity ...
We focus on proving **asymptotic stability for autonomous systems**:

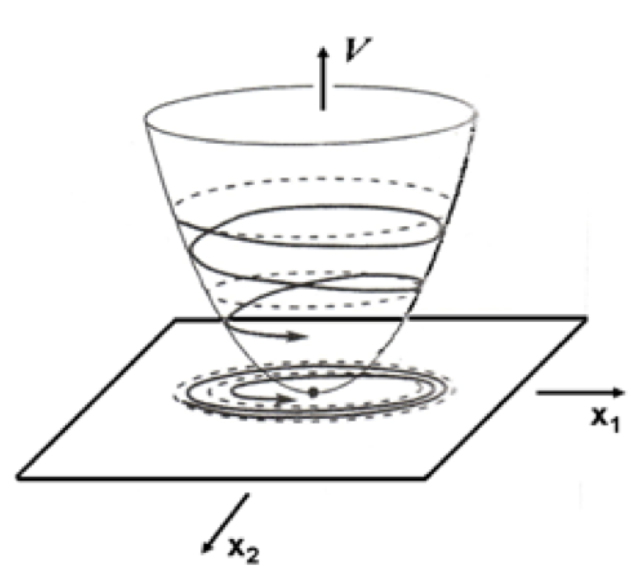
$$\dot{x} = f(x).$$

Lyapunov theory

The system $\dot{x} = f(x)$ is *stable in the sense of Lyapunov* if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that $\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq 0$.

The system is *locally asymptotically stable* (LAS) if it is stable in the sense of Lyapunov and if there exists a scalar $\hat{\delta} > 0$ such that $\|x(0)\| < \hat{\delta} \Rightarrow \lim_{t\to\infty} x(t) = 0$.

The system *globally asymptotically stable* (GAS) if it is stable in the sense of Lyapunov and $\lim_{t\to\infty} x(t) = 0 \; \forall x(0)$.



It is well known that the system is LAS if there exists a continuously differentiable (Lyapunov) function $V : \mathbb{R}^n \to \mathbb{R}$ that vanishes at the origin and satisfies $V(x) > 0$ and $-\langle \nabla V(x), f(x)\rangle > 0$ for all $x \in S \setminus \{0\}$, where S is a neighborhood of the origin.

Moreover, if V is in addition radially unbounded (i.e., satisfies $V(x) \to \infty$ when $\|x\| \to \infty$) and if $S = \mathbb{R}^n$, then the origin is GAS.

How to find such a function V?

- When are algebraic techniques helpful?
- What are some of their limitations?

SOS techniques to the rescue

Idea: Replace positivity constraint with SOS constraint. A polynomial $p(x_1, \ldots, x_n)$ is SOS (sum of squares) if it can be decomposed as

$$p(x) = \sum_{i=1}^{k} q_i^2(x), \quad \text{where } q_i \text{ are polynomials.}$$

SOS decompositions are algorithmically tractable.

$$V(x) \text{ and } -\dot{V}(x) \text{ SOS} \implies V(x) \text{ and } -\dot{V}(x) \geq 0 \implies \text{Stability}$$

Example:

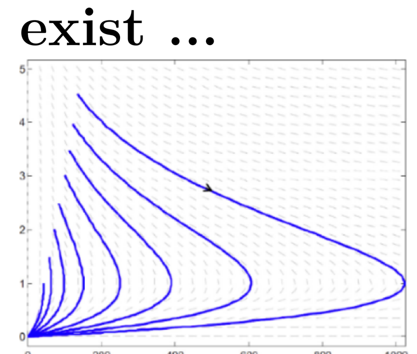
$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \begin{pmatrix}-x^3 - xz^2\\ -y - x^2 y\\ -z - \frac{3z}{z^3+1} + 3x^2 z\end{pmatrix}.$$

Couple of lines of code in SOSTOOLS, YALMIP...

$$V = 4.1068x^2 + 5.5489y^2 + 1.7945z^2.$$

Limitations

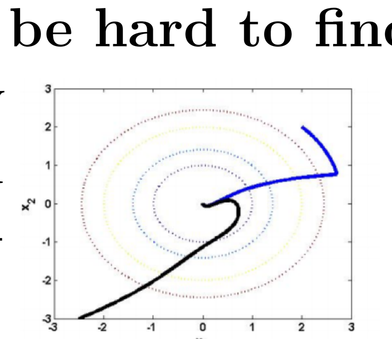
Polynomial Lyapunov functions may not exist ...



$$\begin{aligned}\dot{x} &= -x + xy\\ \dot{y} &= \quad -y\end{aligned}$$ is GAS, but has no polynomial Lyapunov function of any degree [Ahmadi, Krstic, Parrilo]!

... or they can exist but be hard to find

$V(x) = x_1^2 + x_2^2$ is a Lyapunov function, but SOS fails to find any degree 2 Lyapunov function.



A complexity theoretic limitation

Testing asymptotic stability of cubic vector fields is strongly NP-hard [Ahmadi]. Unless P=NP, small-sized algebraic certificates of stability do not exist for some stable systems.

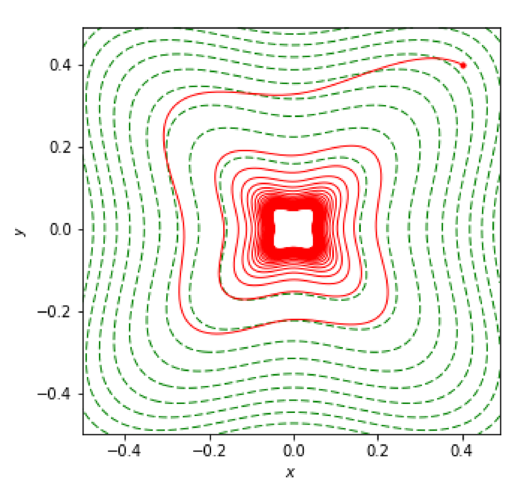
Theorem 1: A more severe limitation

The polynomial dynamical system

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} := f(x,y) = \begin{pmatrix}-2y(-x^4 + 2x^2y^2 + y^4)\\ 2x(x^4 + 2x^2y^2 - y^4)\end{pmatrix} - (x^2+y^2)\begin{pmatrix}2x(x^4 + 2x^2y^2 - y^4)\\ 2y(-x^4 + 2x^2y^2 + y^4)\end{pmatrix}$$

is GAS but does not admit an analytic Lyapunov function even locally.

Proof idea



Consider the rational function $W(x) = \frac{x^4+y^4}{x^2+y^2}$. The function f can written as $f = R\nabla W - (x^2+y^2)\nabla W$, where R is the rotation matrix by $\frac{\pi}{2}$ radians.

This proves that the system is GAS, as W is a corresponding Lyaponuv function.

Near the origin, the dominating term is $R\nabla W$, so the trajectories are close to the level sets of W. Therefore, any analytic Lyapunov function has to be constant on these levels sets, which is impossible.

Theorem 2: A converse result for homogeneous systems

A homogeneous dynamical system $\dot{x} = f(x)$ is asymptotically stable iff it admits a rational Lyapunov function of the form

$$V(x) = \frac{p(x)}{(\sum x_i^2)^r}, \quad \text{where } p \text{ is a polynomial.}$$

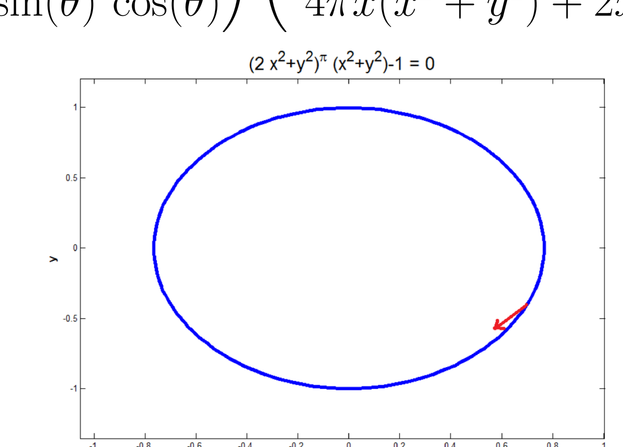
Furthermore, when f is polynomial, V can be found by SOS programming.

Proof idea

- A stable homogeneous system f has a homogeneous Lyapunov functions V [Rosier].
- Use Bernstein polynomials to Approximate V by a polynomial p s.t. $p \approx V$ and and $\nabla p \approx \nabla V$ on the unit cube.
- Homogenize p as $R(x) := \|x\|^2 p(\frac{x}{\|x\|})$ for the approximation to hold on $\mathbb{R}^n$.

The degree of the rational function can be arbitrarily high

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{pmatrix}\begin{pmatrix}-2\pi y(x^2+y^2) - 2y(2x^2+y^2)\\ 4\pi x(x^2+y^2) + 2x(2x^2+y^2)\end{pmatrix}$$



GAS for $\theta \in (0, \pi)$, so a rational Lyapunov function V_θ exists. The degree of the numerator however has to diverge, otherwise, the limit $V_0 := \lim_{\theta\to 0} V_\theta$ would be constant on the trajectories for $\theta = 0$, but in this case the trajectories are non-algebraic periodic orbits.