Study Notes on Matrices & Determinants for GATE 2017

Matrices and Determinates are undoubtedly one of the most scoring and high yielding topics in GATE.

At least 3-4 questions are always anticipated from Matrices and Determinants making it one of the easiest and high-yielding topics in GATE.

In this post, we will provide insights into the Basics of Matrices and Determinants along with the important Shortcuts and Tricks.

Introduction to Matrices & Determinants

I. Matrix

A Matrix is referred to as an ordered rectangular array of numbers.

Matrix ‘A’ containing m rows and n columns is denoted by

\[ A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{n1} & \cdots & a_{mn} \end{pmatrix} \]

In more Compact form, A is represented by

\[ A = (a_{ij}) \]

II. Classification of Matrices

- **Row Matrix**: A Matrix containing a single row is referred to as a Row Matrix.
  For instance: [1 3, 5, 7]

- **Column Matrix**: A Matrix containing a single column is referred to as a Column Matrix.
- **Square Matrix**: A Matrix in which the Number of Rows are equal to the Number of Columns is referred to as a Square Matrix.

**Trace of a Matrix**: In a square matrix, the sum of all elements lying along the Principal Diagonal is referred to as the Trace of the Matrix.

- **Diagonal Matrix**: A Matrix whose all elements except those in the principal diagonal are zero is referred to as a Diagonal Matrix.

- **Identity Matrix**: A Diagonal Matrix of order n having unity for all its elements is referred to as an Identity Matrix

### III. Operations on Matrices

- **Addition & Subtraction of Matrices**
  The condition for addition and subtraction is that the two matrices must have the same number of rows and columns.

**Addition**
Two matrices A and B can be added by adding up their corresponding elements.

**For Instance**: If A and B are two matrices denoted by:

\[
A = \begin{pmatrix}
1 & 2 & 3 \\
1 & 0 & 2
\end{pmatrix}
and
B = \begin{pmatrix}
2 & 1 & 2 \\
1 & 0 & 3
\end{pmatrix}
\]

\[
A + B = \begin{pmatrix}
1 & 2 & 3 \\
1 & 0 & 2
\end{pmatrix} + \begin{pmatrix}
2 & 1 & 2 \\
1 & 0 & 3
\end{pmatrix} = \begin{pmatrix}
3 & 3 & 5 \\
2 & 0 & 5
\end{pmatrix}
\]
**Subtraction**: Two matrices $A$ and $B$ can be subtracted by subtracting their corresponding elements.

*For Instance:*

$$A - B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

- **Matrix Multiplication**

Multiplication of Two Matrices can only be performed when the number of columns in the first matrix is equal to the number of rows in the second matrix.

**IV. Transpose of a Matrix**

The Transpose of a Matrix is obtained by interchanging its rows and columns. Transpose of a Matrix $A$ is denoted by $A^T$.

*For Instance:*

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

**V. Determinant of a Matrix**

Determinant of a Matrix plays a significant role in computing the inverse of the matrix and in also solving the system of linear equations.

- **Determinant of a $2 \times 2$ Matrix**

Let $A$ denote a $2 \times 2$ matrix whose elements are given by

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$
Then the Determinant of the Matrix A is denoted by

\[ \det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \]

- **Determinant of a 3 X 3 Matrix**

The determinant of a 3 x 3 Matrix is computed as follows:

\[ \det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \]

VI. **Inverse of a Matrix**

The Inverse of a general \( n \times n \) Matrix can be computed using the following formula:

\[ A^{-1} = \frac{\text{adj}(A)}{\det(A)} \]

VII. **Orthogonal Matrix**

Matrix A is said to be an Orthogonal Matrix if

\[ A^T A = I \]

VIII. **Rank of a Matrix**

The Rank of a Matrix is referred to as the Maximum Number of Linearly Independent Row vectors in the Matrix or the Maximum number of linearly independent column vectors in the Matrix.
IX. Eigen Values and Eigen Vectors

Let \( P \) denote an \( n \times n \) Matrix. Then, the number \( \lambda \) is referred to as the Eigenvalue of \( P \) if there exists a non-zero vector \( V \) such that the following condition holds:

\[
P V = \lambda V
\]

In this case, Vector ‘\( V \)’ is referred to as the Eigenvector of \( P \) corresponding to \( \lambda \).

Computing Eigen Values and Eigen Vectors

We can rewrite the \( PV = \lambda V \) as

\[
(P - \lambda I)V = 0
\]

Where \( I \) is the \( n \times n \) Identity Matrix

In order that Non-zero Vector \( V \) satisfies this equation, \( P - \lambda I \) must not be Invertible.

This implies that Determinant of \( P - \lambda I \) must be equal to 0.

We call \( s(\lambda) = \det (P - \lambda I) \) is the characteristic polynomial of \( P \).

Then, Eigenvalues of \( P \) are the roots of the characteristic polynomial of \( P \).
X. **Solving System of Linear Equations using Matrices**

A system of linear equations is referred to as a set of equations containing ‘n’ equations and ‘n’ unknowns.

A System of Linear Equations is represented as:

\[
\begin{align*}
\alpha_{11}x_1 + \alpha_{12}x_2 + \ldots + \alpha_{1n}x_n &= \beta_1 \\
\alpha_{21}x_1 + \alpha_{22}x_2 + \ldots + \alpha_{2n}x_n &= \beta_2 \\
&\vdots \\
\alpha_{n1}x_1 + \alpha_{n2}x_2 + \ldots + \alpha_{nn}x_n &= \beta_n
\end{align*}
\]

Where

\(x_1, x_2, \ldots, x_n\) denote the unknowns and the coefficients are assumed to be given.

System of Linear Equations is Matrix Form is represented as:

\[
\begin{pmatrix}
\alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\
\alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n1} & \alpha_{n2} & \ldots & \alpha_{nn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
=
\begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_n
\end{pmatrix}
\]
XI. Method to Solve System of Linear Equations

• Cramer’s Rule

Consider a system of linear equations denoted by
\[ a_1x + b_1y + c_1z = d_1 \]
\[ a_2x + b_2y + c_2z = d_2 \]
\[ a_3x + b_3y + c_3z = d_3 \]

Let

\[
\Delta = \begin{vmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3 \\
\end{vmatrix},
\Delta x = \begin{vmatrix}
  d_1 & b_1 & c_1 \\
  d_2 & b_2 & c_2 \\
  d_3 & b_3 & c_3 \\
\end{vmatrix},
\Delta y = \begin{vmatrix}
  a_1 & d_1 & c_1 \\
  a_2 & d_2 & c_2 \\
  a_3 & d_3 & c_3 \\
\end{vmatrix},
\Delta z = \begin{vmatrix}
  a_1 & b_1 & d_1 \\
  a_2 & b_2 & d_2 \\
  a_3 & b_3 & d_3 \\
\end{vmatrix}
\]

Then

\[ x = \frac{\Delta x}{\Delta}, y = \frac{\Delta y}{\Delta}, z = \frac{\Delta z}{\Delta} \]