Limits, Continuity and Differentiability - GATE Study Material in PDF

When dealing with Engineering Mathematics, we are constantly exposed to Limits, Continuity and Differentiability. These concepts in calculus, first proposed separately by Isaac Newton and Gottfried Leibniz, have permeated every walk of life – from space sciences to sewage management. But for a student of Engineering, these concepts form the bedrock of all their curriculum. They are especially important for GATE EC, GATE EE, GATE CS, GATE CE and GATE ME. They also appear in other exams like BSNL, BARC, IES, DRDO etc.

You may download these free GATE 2018 Notes in PDF so that your preparation is made easy and you ace your paper. You may also want to read the following articles on Engineering Mathematics.

Recommended Reading List -

Types of Matrices

Properties of Matrices

Rank of a Matrix & Its Properties

Solution of a System of Linear Equations

Eigen Values & Eigen Vectors

Linear Algebra Revision Test 1

Laplace Transforms

Limits

Suppose f(x) is defined when x is near the number a. (this means that f is defined on some open interval that contains a, except possibly at 'a' itself.)

Then we can write $\lim_{x \to a} f(x) = L$

And we can say, "the limit of f(x), as x approaches a, equals L"

An alternative notation for $\lim_{x \to a} f(x) = L$ is $f(x) \to L$ as $x \to a$

Example 1:

Find the value $\lim_{x \to 1} \frac{x-1}{x^2-1}$

Solution:

Here $f(x) = \frac{x-1}{x^2-1}$ is not defined at $x = 1$ but that does not matter because the

definition of $\lim_{x \to a} f(x)$ says that we consider values of x that are close to 'a' but not

equal to 'a'.

So, from the above two tables we can say that

$$\lim_{x \to 1} \frac{x-1}{x^2-1} = \lim_{x \to 1} \frac{x-1}{(x-1)(x+1)}$$

$$\therefore \lim_{x \to 1} \frac{1}{x+1} = 0.5$$

Example 2:

Find the value of $\lim_{x\to 0}\frac{\sin x}{x} = ?$

Solution:

The expansion of sin x according to Taylor series is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ \ldots\ldots$$

$$f(x) = \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \ \ldots\ldots$$

$$\lim_{x\to 0} f(x) = 1$$

Note:

1. $\lim_{x\to a} f(x) = L$, if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$

2. sin x is a bounded function and it oscillates between -1 and 1 i.e. $-1 \le \sin x \le 1$

Limit Laws

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist then

1. $\lim_{x\to a}[cf(x)] = c\left[\lim_{x\to a} f(x)\right]$ where c is constant.
2. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) . \lim_{x\to a} g(x)$
3. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
4. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is positive integer.
5. $\lim_{x\to a} c = c$
6. $\lim_{x\to a} x = a$
7. $\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} f(x)^{\lim_{x\to a} g(x)}$

8. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$

Standard limit Values

1. $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$
2. $\lim_{x \to 0} \frac{e^x - 1}{x} = \log_e e = 1$
3. $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$
4. $\lim_{x \to 0} \frac{\sin x}{x} = 1$
5. $\lim_{x \to 0} \frac{\tan x}{x} = 1$
6. $\lim_{x \to 0} (1 + ax)^{\frac{b}{x}} = e^{ab}$
7. $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^{bx} = e^{ab}$
8. $\lim_{x \to 0} \left(\frac{a^x + b^x}{2}\right)^{\frac{1}{x}} = \sqrt{ab}$
9. $\lim_{x \to \infty} a^x \sin \frac{b}{a^x} = b$
10. $\lim_{x \to \infty} a^x \tan \frac{b}{a^x} = b$
11. $\lim_{x \to 0} \frac{\sinh x}{x} = 1$
12. $\lim_{x \to 0} \frac{\tanh x}{x} = 1$
13. $\lim_{x \to 0} \frac{\sin^{-1}(x)}{x} = 1$
14. $\lim_{x \to 0} \frac{\tan^{-1}(x)}{x} = 1$

15. $\lim_{x\to 0}\frac{1-\cos ax}{x^2}=\frac{a^2}{2!}$

16. $\lim_{x\to 0}\frac{1-\cos ax}{x^4}=\frac{a^4}{4!}$

Indeterminate Forms

The following forms are to be considered as Indeterminate forms.

$\frac{0}{0},\frac{\infty}{\infty},0\times\infty,\infty-\infty,0^0,1^\infty,\infty^0$

If the functions are in the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then to find limit we need to apply L′ hospital's rule.

$$\text{i.e.} \lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)}=\frac{\lim_{x\to a} f''(x)}{\lim_{x\to a} g''(x)}\dots\dots\dots.= L$$

Example 3:

Find the value of $\lim_{x\to 0}\frac{\sin x-x}{x^2}$

Solution:

It is in $\frac{0}{0}$ form

So, to solve the above problem we need to apply L hospital rule

$$\frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)}=\frac{\cos x-1}{2x}=\left(\frac{0}{0}\right)\text{ Form}$$

Again apply L Hospital Rule then

$$\frac{\lim_{x\to a} f''(x)}{g''(x)} = \lim_{x\to 0} \frac{-\sin x}{2} = 0$$

Note:

1. If the functions are in the form of $(0 \times \infty)$ then reducing them into $\frac{\infty}{\infty}$ (or) $\frac{0}{0}$ and then apply L' hospital rule.
2. If the limit value is in the form of (∞ - ∞) then we need to follow this procedure to find the limit value.

$$\text{i. e.} \lim_{x\to a}\big(f(x) - g(x)\big) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

3. If f(x) = ∞ and g(x) = ∞   as $x \to a$

$$\text{Then } \lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a}\left[\frac{1}{g(x)} - \frac{1}{f(x)}\right] \div \frac{1}{f(x)g(x)}$$

Then apply L' hospital form.

Example 4:

Find the value of $\lim_{x\to\frac{\pi}{2}}(\sec x - \tan x)$

Solution:

$$\lim_{x\to\frac{\pi}{2}} (\sec x - \tan x) = \lim_{x\to\frac{\pi}{2}}\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{(1-\sin x)}{\cos x}$$

Now it is in $\left(\frac{0}{0}\right)$ form L – Hospital Rule gives $= \lim_{x\to\frac{\pi}{2}}\frac{-\cos x}{-\sin x} = 0$

Example 5:

Find the value of $\lim_{x\to 0} x^x$

Solution:

It is in the form of 0^0 and the following procedure is used to find the value of limit.

Let $y = x^x$

$\log y = x \log x$

$$\lim_{x\to 0} \log y = \lim_{x\to 0} (x \log x)$$

$$\lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log x}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0} -x = 0$$

$\log y = 0 \quad \Rightarrow \quad y = x^x = e^0 = 1$

Example 6:

Find the value of $\lim_{x\to\frac{\pi}{2}} (\tan x)^{\cos x}$

Solution:

It is in the form of (∞^0)

Let $y = \tan x^{\cos x}$

$$\therefore \lim_{x\to\frac{\pi}{2}} \log y = \lim_{x\to\frac{\pi}{2}} \cos x \log \tan x$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\log \tan x}{\sec x}, \text{Now, it is in the form of } \left(\frac{\infty}{\infty}\right)$$

$$\therefore \lim_{x\to\frac{\pi}{2}} \frac{1}{\tan x} \cdot \frac{\sec^2 x}{\sec x \tan x}$$

$$\lim_{x\to\frac{\pi}{2}} \frac{\sec x}{\tan^2 x}$$

Again applying L-Hospital’s rule gives,

$$\lim_{x\to\frac{\pi}{2}} \frac{\sec x \tan x}{2\tan x \sec^2 x}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{2} = 0$$

$\therefore y = e^0 = 1$

Continuity

A function f is said to be continuous at a number 'a' if $\lim_{x\to a} f(x) = f(a)$

Notice that above definition requires three things if f is constant at a:

1. f(x) is defined
2. $\lim_{x\to a} f(x)$ exist
3. $\lim_{x\to 0} f(x) = f(a)$

Example 7:

For what value of k. the given function is continuous?

$$f(x) = \left.\begin{array}{lll}(1+kx)^{\frac{1}{x}} & \text{when} & x \neq 0 \\ e^3 & \text{when} & x = 0\end{array}\right\}$$

Solution:

$$\lim_{x\to 0} f(x) = e^{kx \times \frac{1}{x}} = e^k$$

But, $f(0) = e^3$

Since, the function f(x) is continuous $\therefore k = 3$

Differentiability

Suppose f is a real function and c is a point in its domain. The derivative of the function f at c is defined by

$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$, provided the limit of the function exists.

Derivative of function f at c is denoted by $f'(c)$ or $\frac{d}{dx}(f(x))\Big|_{x=c}$

The process of finding derivative of a function is called differentiation.

Note:

1. If $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ does not exist then we can say that f is not differentiable at c.

2. In other words we say that a functions f is differentiable at a point c in its domain if both $\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h}$ and $\lim_{h\to 0^-} \frac{f(c-h)-f(c)}{-h}$ are finite and equal.

Standard Rules of Differentiation

1. $(U \pm V)' = U' \pm V'$
2. $(UV)' = U'V + UV'$
3. $\left(\frac{U}{V}\right)' = \frac{VU'-UV'}{V^2}$

Example 8:

The function $f(x) = x\sin\frac{1}{x}$ at $x = 0$ is differentiable or not differentiable?

Solution:

Right Hand Limit (R):

$f'(c+) = \lim_{x\to 0+h} \frac{f(0+h)-f(0)}{h} = \frac{h\sin\frac{1}{h}}{h} = \sin\frac{1}{h}$

Left Hand Limit (L):

$$f'(c-) = \lim_{x\to 0-h} \frac{f(0-h)-f(0)}{-h} = \frac{-h\sin\left(-\frac{1}{h}\right)}{-h}$$

$$= -\sin\frac{1}{h}$$

$$f'(c^+) \neq f'(c^-)$$

Hence, it is not differentiable.

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