Routh Hurwitz Stability Criteria - GATE Study Material in PDF

Now that we know the Concept of Stability in Linear Time Invariant Systems, we can move on to the next concept in Stability – Routh Hurwitz Stability Criteria. This forms one of the most important topics in Control Systems. You can download this free GATE Study Material in PDF.

Recommended Reading –

Stability of Control Systems

Earlier, we studied about the concepts and condition for determining the stability of a system. However, we need to find the roots of the characteristic equation i.e. poles to determine the stability of the system. In the analysis, the characteristic equations are mostly large and complex. Hence it is difficult to simplify them into roots (like those of order 4 and above). For such situations, Routh Hurwitz Method provides an easy and quick method to determine the stability without the need to disintegrate the characteristic equation.

Routh Hurwitz Stability Criteria

Routh Hurwitz Stability Criterion is based on ordering the coefficients of the characteristic equation into an array, also known as Routh Array.

Suppose the characteristic equation of a control system is given as:

$q(s) = a_0s^n + a_1s^{n-1} + a_2s^{n-2} + \ldots + a_{n-1}s + a_n = 0$

Now, from the given equation, we will form Routh Array as shown below:

$$
\begin{array}{c|l}
s^n & a_0\ a_2\ a_4\ a_6\ -\ -\ - \\
s^{n-1} & a_1\ a_3\ a_5\ a_7\ -\ -\ - \\
s^{n-2} & b_1\ b_2\ b_3\ b_4\ -\ -\ - \\
s^{n-3} & c_1\ c_2\ c_3\ -\ -\ -\ - \\
s^{n-4} & d_1\ d_2\ -\ -\ -\ -\ - \\
s^{n-5} & . \\
\vdots & . \\
s^2 & . \\
s^1 & . \\
s^0 & .
\end{array}
$$

$a_0, a_1, ..., a_n$ coefficients are taken from the equation and arranged as shown.

Other elements are calculated from these element. Coefficients $b_1, b_2, b_3, ..., b_n$ are calculated as

$$b_1 = \frac{a_1a_2 - a_0a_3}{a_1}$$

$$b_2 = \frac{(a_1a_4 - a_5a_0)}{a_1}$$

$$b_3 = \frac{(a_1a_6 - a_7a_0)}{a_1}$$

This process is continued till we get zero in the row with b coefficients.

Similarly, c coefficients and d coefficients are calculated as following:

$$c_1 = \frac{b_1a_3 - b_2a_1}{b_1}$$

$$c_2 = \frac{b_1a_5 - a_1b_3}{b_1}$$

$$d_1 = \frac{c_1b_2 - b_1c_2}{c_1}$$

$$d_2 = \frac{c_1b_3 - b_1c_3}{c_1}$$

In this process, the missing terms are considered zero and elements of any row can be divided by a positive number to simplify the calculation.

Definition of Routh Hurwitz Stability Criteria

Now, the Routh stability Criteria is given as:

"For a system to be stable, it is necessary and sufficient that each term of first column of Routh Array formed of its characteristic equation be positive if $a_0>0$. If this condition is not met, the system is unstable and number of sign changes of the terms of the first column of the Routh Array corresponds to the number of roots of the characteristic equation in the right half of the s-plane".

Examples of Routh Hurwitz Stability Criteria

We will understand the usage of Routh Hurwitz Criteria through following examples.

Example 1: The characteristic equation of a system is given below. Determine the stability of the system.

$s^4 + 4s + 16s^2 + 10s^2 + 6 = 0$

Solution:

Applying Routh Hurwitz Criteria and forming Routh array, we get

$$\begin{array}{llll}
s^4 & 1 & 16 & 5 \\
s^3 & 4 & 10 & 0\ (for\ missing\ term) \\
s^2 & \frac{16\times4-10\times1}{4} = 13.5 & \frac{5\times4-0\times1}{4} = 5 & \\
s^1 & \frac{13.5\times10-5\times4}{13.5} = 8.52 & 0 & \\
s^0 & 5 & &
\end{array}$$

In the Routh array formed, if we see in first column; all the elements are positive. There is no sign change. Hence the system in question is stable.

Example 2: The characteristic equation of a system is given below. Determine the stability of the system

$4s^4 + 8s^3 + 2s^2 + 10s + 3 = 0$

Solution:

Applying Routh Hurwitz Criteria and forming Routh array. We get,

$$\begin{array}{llll}
s^4 & 4 & 2 & 3 \\
s^3 & 8 & 10 & 0 \\
s^2 & \frac{8\times2-4\times10}{8} = -3 & \frac{8\times3-0\times4}{8} = 3 & \\
s^1 & \frac{-3\times10-3\times8}{-3} = 18 & & \\
s^0 & 3 & &
\end{array}$$

In the first column of the Routh Array formed above, there is one negative element. Also, there are two sign changes in first column. First is from 8 to -3 and second is from -3 to 18. Hence the system is questions is unstable and out of 4 poles, 2 are in the right half of s-plane.

Now, apart from determining the stability, Routh Criteria can also be used for tuning the variable parameters to keep the system in the stable region. This can be understood from the following example.

Example 3: The characteristic equation of a system is given as follows

$s^3 + 3s^2 + 7s + k = 0$

Find the range of values of k for which the system would be stable.

Solution:

In the given system, k is an unknown parameter. Now, forming Routh array from the given equation

$$\begin{array}{ccc} s^3 & 1 & 7 \\ s^2 & 3 & k \\ s^1 & \frac{21-k}{3} & 0 \\ s^0 & k & 0 \end{array}$$

(Try to form above array yourself)

Now, in the array formed; if the system is to be stable then all the elements in first column need to be positive

$\therefore 21 - k > 0$ & $k > 0$

$k < 21$ & $k > 0$

$\Rightarrow 0 < k < 21$ for the system to be stable.

The equations might be different but will get solved using this one rule only.

Special Cases of Routh Hurwitz Stability Criteria

However, in this criteria there are some special conditions in which some assumptions are needed to be made. These special cases are mentioned as follows:

Case I: When the first term in any row of the Routh array is zero while rest of the row has at least one non-zero term.

Because of this zero term, the terms in the next row become infinite and Routh's test breaks down. To overcome this difficulty, substitute a small positive number ε for zero and proceed to evaluate the rest of Routh Array. Then check the signs of the first column of the array by substituting ε→0.

This can be better understood by following example.

Example 4: The characteristic equation of a system is given as

$s^5 + s^4 + 3s^3 + 3s^2 + 2s + 5 = 0$

Determine whether the system is stable or not.

Solution:

Applying Routh Hurwitz Criteria and forming Routh Array, we get

$$\begin{array}{llll} s^5 & 1 & 3 & 2 \\ s^4 & 1 & 3 & 5 \\ s^3 & 0 \rightarrow \epsilon & -3 & 0 \\ s^2 & \frac{3\epsilon+3}{\epsilon} & 5 & 0 \\ s^1 & \frac{\frac{-3(3\varepsilon+3)}{\epsilon}-5\epsilon}{\frac{3\epsilon+3}{\epsilon}} & & \\ s^0 & 5 & & \end{array}$$

Now, if we examine the elements of first column.

$s^3 \rightarrow \varepsilon > 0$

$s^2 \rightarrow \frac{3\epsilon+3}{\epsilon} = 3 + \frac{3}{\epsilon} > 0$

$s^1 \rightarrow \frac{-9\epsilon-9-5\epsilon^2}{3\epsilon+3} < 0 \text{ (as } \epsilon > 0)$

Since there is a sign change at s^1 row, hence the system is unstable and having two poles in right half of s-plane due to two sign changes.

Case II: When all the elements in any one row of the Routh Array are zero.

In this case there are symmetrically located roots in the s-plane. There can be pair of real roots with opposite signs and /or pair of conjugate roots on the imaginary axis and/or complex conjugate roots forming quadrature in the s-plane. The polynomial whose coefficients are the elements of the row just above the row of zeros in the Routh array is called an auxiliary polynomial. This polynomial gives the number and location of root pairs of the characteristic equation which are symmetrically located in the s-plane. The order of the auxiliary polynomial is always even. We will be able to understand this case better with the following example.

Example 5: The characteristic equation of a system is given as follows. Comment on the stability of the system.

$s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$

Solutions:

The Routh array formed is

s^6	1	8	20	16
s^5	2	12	16	0
s^4	2	12	16	0
s^3	0	0	0	0
s^2				
s^1				
s^0				

As we see, s^3 row is completely zero. Hence the auxiliary polynomial is formed from s^4 row, i.e. $2s^4 + 12s^2 + 16 = 0$

Or $s^4 + 6s^2 + 8 = 0$

Differentiating the polynomial w.r.t s, we get

$4s^3 + 12s = 0$

The zeros in the s^3 row are now replaced by the coefficients of derivative of auxiliary polynomial. The Routh array now formed will be

s^6	1	8	20	16
s^5	2	12	16	
s^4	1	6	8	
s^3	4	12	0	
s^2	3	8		
s^1	1/3			
s^0	8			

In above array, there is no change of sign. Hence the system will be marginally or limitedly stable.

Also, if we solve and find the roots of auxiliary polynomial

$s^4 + 6s^2 + 8 = 0$

The roots are $s = \pm j\sqrt{2}$ and $s = \pm\, j2$

These two pair of roots are also among the roots of given characteristic equation.

Thus, above were the special cases for Routh Hurwitz stability Criteria. From point of view of GATE exam, questions from this topic are often asked in Control Systems.

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