ECON 2723, Asset Pricing, Section 6

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Admin

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Topics to cover

Chapter 9 (Intertemporal Portfolio Choice)

- Intertemporal Hedging Demand
- Hedging Interest Rates
- Hedging Risk Premium
- Two- and Three Beta CAPM

9.1 Myopic Portfolio Choice

- Consider an investor maximizing his wealth in K periods W_{t+k}.
- This investor is **myopic** if her utility is maximized with assets that maximize W_{t+1}.

- We talked about some sufficient conditions for myopic portfolio choice. These conditions are either on **the DGP of returns** or **the investor's utility**:
  - If you allow rebalancing at every period t,
    - Log utility ($\gamma = 1$) is sufficient.
    - Power utility ($\gamma \neq 1$) + iid returns is sufficient.
  - If you do not allow rebalancing (weird), Epstein-Zin + lognormal returns is sufficient.

- For more details, see *Myopic Portfolio Choice*, Campbell and Viceira

9.2 Intertemporal Hedging

- Consider "a simple example" (9.2.1)
  - **Investor**: Power utility investor maximizing his wealth in K periods.
  - **Returns DGP**: Log-normal return on single risky asset, not necessarily independently nor identically distributed over time.
  - **Rebalancing**: Yes

- Approximations from Chapter 2 yield

$$\max_{\alpha_t} E_t r_{p,K,t+K} + \frac{1}{2}(1-\gamma) var_t(r_{p,K,t+K}) \qquad \underbrace{\Longleftrightarrow}_{\pm\{E_t(r_{p,t+1})+\frac{1}{2}var(r_{p,t+1})\}}$$

$$\Longleftrightarrow \max_{\alpha_t} E_t[r_{p,t+1}] + \frac{1}{2}(1-\gamma) var_t(r_{p,t+1}) + \left[E_t r_{p,K,t+K} - E_t[r_{p,t+1}]\right] + \frac{1-\gamma}{2}\left[var_t(r_{p,K,t+K}) - var_t(r_{p,t+1})\right]$$

- The maximization problem is over the short-term portfolio choice α_t
  - Take future choices α_{t+k} as given, because portfolio choice with scale-independent preferences is forward looking and independent of past returns.

- The only way for α_t to be different than the one-period solution is for it to affect the $t+k$ terms.

Intertemporal Hedging

- We can decompose K period return into $r_{p,K,t+K} = r_{p,t+1} + r_{p,K-1,t+K}$ to simplify the second part of the last expression...

$$\iff \max_{\alpha_t} E_t[r_{p,t+1}] + \frac{1}{2}(1-\gamma)var_t(r_{p,t+1}) + E_t r_{p,K-1,t+K} + \frac{1-\gamma}{2}\left[var_t(r_{p,t+1} + r_{p,K-1,t+K}) - var_t(r_{p,t+1})\right]$$

$$\iff \max_{\alpha_t} E_t[r_{p,t+1}] + \frac{1}{2}(1-\gamma)var_t(r_{p,t+1}) + \underbrace{E_t r_{p,K-1,t+K}}_{\text{Doesn't depend on } \alpha_t} + \frac{1-\gamma}{2}\left[\underbrace{var_t(r_{p,K-1,t+K})}_{\text{Doesn't depend on } \alpha_t} + 2cov_t(r_{p,t+1}, r_{p,K-1,t+K})\right]$$

- Difference between K-period and 1-period expected portfolio returns is in fact unaffected by α_t, due to the independence of portfolio choice from past returns and the linearity of the expectation operator

- However, portfolio choice today affects the long-run riskiness of the portfolio.

Intertemporal Hedging

- Now subtract risk free rate $r_{f,t+1}$, and apply portfolio approximation from Chapter 2 to get

$$\iff \max_{\alpha_t} \left[E_t r_{p,t+1} - r_{f,t+1}\right] + \frac{1}{2}(1-\gamma) var_t(r_{p,t+1}) \\ + (1-\gamma) cov_t(r_{p,t+1}, r_{p,K-1,t+K})$$

$$\iff \max_{\alpha_t} \alpha_t (E_t r_{t+1} - r_{f,t+1}) + \frac{1}{2}\alpha_t(1-\alpha_t)\sigma_t^2 + \frac{1}{2}(1-\gamma)\alpha_t^2\sigma_t^2 \\ + (1-\gamma) cov_t(r_{p,t+1}, r_{p,K-1,t+K})$$

Intertemporal Hedging

- First order condition

$$
\begin{aligned}
0 =& (E_t r_{t+1} - r_{f,t+1}) + \frac{1}{2}(1 - 2\alpha_t)\sigma_t^2 + (1-\gamma)\alpha_t\sigma_t^2 \\
& + (1-\gamma)\frac{d}{d\alpha_t} cov_t(r_{p,t+1}, r_{p,K-1,t+K})
\end{aligned}
$$

$$
\implies \alpha_t = \underbrace{\frac{E_t r_{t+1} - r_{f,t+1} + \frac{1}{2}\sigma_t^2}{\gamma\sigma_t^2}}_{\text{standard term from chapter 2}} - \underbrace{\frac{\gamma - 1}{\gamma\sigma_t^2} \cdot \frac{d}{d\alpha_t} cov_t(r_{p,t+1}, r_{p,K-1,t+K})}_{\text{intertemporal hedging term}}
$$

- **Intertemporal hedging term**:
  - Suppose that investment opportunities at period $t+1$ (expectation of returns going forward) are positively correlated with return at time $t+1$
  - The agent has more wealth when investment opp. are good going forward and less wealth when investment opp. are bad going forward.
  - This increases the volatility of two periods return. In this case, $cov_t \propto \alpha_t$.
  - This is penalized by a conservative investor with $\gamma > 1$ so that the asset weight of risky asset is lower in equation for α_t

Later we will see how investment opportunities affect the marginal value of wealth that drives the distinction between $\gamma > 1$ and $\gamma < 1$.

Hedging Interest Rates

Let's apply the intertemporal portfolio choice to two different problems...

- Now consider (9.2.2):
  - Investor: Epstein-Zin
  - Changing interest rates
  - Constant second moments:
  - Constant risk premium $\Longrightarrow$ returns scale with risk free rates:

$$\mathbb{E}_t[r_{i,t+1}] - r_{f,t+1} + \underbrace{\frac{\sigma_i^2}{2}}_{constant} = constant$$

- In particular, this implies[1]

$$\underbrace{\tilde{h}_{t+1}}_{\text{innovation in expected future wealth returns}} = (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{p,t+1+j}$$

$$= (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{f,t+1+j}$$

[1] Writing equation (9.8) for wealth

Hedging Interest Rates

- Remember the first order condition for the Epstein-Zin utility function (ICAPM, eq. 6.54)[2]

$$E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} = \gamma \overbrace{\underbrace{cov_t(r_{i,t+1}, r_{p,t+1})}_{\text{covariance with wealth portfolio}}}^{\text{traditional CAPM}}$$

$$+ (\gamma - 1)\, \underbrace{cov_t\left(r_{i,t+1}, \overbrace{(E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{f,t+1+j}}^{\text{changing investment opportunities}} \right)}_{\text{covariance with revisions in future expected wealth returns}}$$

- In a simple case of 1 risky asset...

$$cov_t(r_{i,t+1}, r_{p,t+1}) = cov_t(r_{i,t+1}, \alpha_t r_{p,t+1}) = \alpha_t \sigma^2$$

[2] Yes weird, the ψ's go away!

Hedging Interest Rates

$$\stackrel{FOC}{\Longrightarrow} E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma^2}{2} = \gamma \alpha_t \sigma^2 + (\gamma - 1) cov_t \left(r_{i,t+1}, (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{f,t+1+j} \right)$$

$$\Longrightarrow \alpha_t = \underbrace{\frac{1}{\gamma} \frac{E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma^2}{2}}{\sigma^2}}_{\text{Myopic Portfolio Demand}} + \underbrace{\left(1 - \frac{1}{\gamma}\right) \frac{1}{\sigma^2} cov_t \left(r_{i,t+1}, -(E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{f,t+1+j} \right)}_{\text{Interest Rate Hedging Term}}$$

- For a long-term investor, if interest rates go down, that is bad for you, and so you hedge them.
- Because RP is constant, innovations to the risk-free rate drive all future investment opportunities $\Longrightarrow$ demand for hedging future interest rates.

Hedging Interest Rates

$$\alpha_t = \underbrace{\frac{1}{\gamma} \frac{E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma^2}{2}}{\sigma^2}}_{\text{Myopic Portfolio Demand}} + \underbrace{\overbrace{\left(1 - \frac{1}{\gamma}\right)}^{\text{conservative inv.}} \frac{1}{\sigma^2} cov_t \left(r_{i,t+1}, -(E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{f,t+1+j} \right)}_{\text{Interest Rate Hedging Term}}$$

- Note that the two terms are weighted by $\frac{1}{\gamma}$ and $1 - \frac{1}{\gamma}$.
  - Consider the case of an infinitely risk-averse investor with $\gamma \to \infty$. In this case the optimal allocation is **only hedging**

$$\alpha_t = \frac{1}{\sigma^2} cov_t \left(r_{i,t+1}, -(E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{f,t+1+j} \right)$$

  - Infinitely risk averse investor wants to invest in assets that perform well when future interest rates go down.
  - Real perpetuities ("inflation indexed bond that goes on forever", IIP) that go down in price when future interest rates go up offer such a product for this investor.
  - Notice that in the 1 period horizon problem the optimal allocation of an infinitely risk averse investor was to invest zero in the risky asset and everything in the short term risk free asset.

Hedging Risk Premium

- Now consider (9.2.3):
  - Constant risk free rate
  - Single risky asset
  - (**Investor**: Guess a portfolio rules α linear and $\exp(b_0 + b_1 x + b_2 x^2)$ value function. Next slide.)

- The single risky asset with return r_{t+1} is governed by:

$$\text{Unexpected Change: } r_{t+1} = E_t r_{t+1} + u_{t+1} \implies u_{t+1} = r_{t+1} - E_t r_{t+1}$$

$$\text{Risk Premium: } E_t r_{t+1} - r_f + \frac{\sigma^2}{2} = x_t$$

$$\text{Process for RP: } x_{t+1} = \mu + \phi(x_t - \mu) + \eta_{t+1}$$

  - When $\sigma_{u\eta} < 0$, shocks to risk premium and unexpected returns are negatively correlated ("mean reversion")

  - We believe that it is the case empirically.

Hedging Risk Premium: Value Function

- Portfolio choice rule and value functions are

$$\alpha(x) = a_0 + a_1 x$$
$$V(x) \sim b_0 + b_1 x + \underbrace{b_2 x^2}_{(*)}$$

  - $(*)$ When the RP increases...
    1. The expected portfolio return for any given investment in the risky asset increases.
    2. The optimal portfolio weight on the risky asset increases.
  - When $\mu = 0$ then $b_1 = 0$ (just stated in book; proved numerically in CV (1999)) so that the minimum value function is achieved at $x^{min} = 0$.
  - When $\mu > 0$ and $\sigma_{u\eta} < 0$, then $b_1 > 0$ so that minimum of the value function is achieved at $x^{min} = -b_1/2b_2 < 0$.

Hedging Risk Premium: Portfolio Rule

Let's consider the determinants of a_0 and a_1 in $\alpha(x) = a_0 + a_1 x$

$$a_0 = \left(1 - \frac{1}{\gamma}\right)(b_1^* + 2\mu(1-\phi)b_2^*)\left(-\frac{\sigma_{\eta u}}{\sigma_u^2}\right),$$

$$\begin{aligned} b_0 &= (1-\psi)b_0^* + \psi \log(1-\delta) \\ b_1 &= (1-\psi)b_1^* \\ b_2 &= (1-\psi)b_2^*, \quad b_2^* > 0 \end{aligned}$$

Intercept.

- When $\mu = 0$, (then $b_1^* = 0$), then $a_0 = 0$:
  - The slope of the value function is zero at $x = 0 \implies$ investor indifferent to mg changes in RP.
  - No incentive to hedge changes in the RP.
- When $\gamma > 1$, $\mu > 0$, and $\sigma_{u\eta} < 0$, then $a_0 > 0$.
  - The agent holds the risky asset even when $x = 0$.
  - At $x = 0$, the value function is *increasing* at the RP, since $x^{min} < 0$.
  - It is bad for the agent to go into $x_t < 0$ region.
  - Since $\sigma_{u\eta} < 0$ the asset has high return when x_t goes down and, therefore, is a hedge against a declining risk premium

Hedging Risk Premium: Portfolio Rule

Let's consider the determinants of a_0 and a_1 in $\alpha(x) = a_0 + a_1 x$

$$a_1 = \frac{1}{\gamma \sigma_u^2} + \left(1 - \frac{1}{\gamma}\right)(2\phi b_2^*)\left(-\frac{\sigma_{\eta u}}{\sigma_u^2}\right)$$

Slope.

- Under $\gamma > 1$ and $\sigma_{u\eta} < 0$, the slope is higher than that of the myopic investor $(1/\gamma\sigma^2)$.
- Long-term investors should **time the market more aggressively** than short term investors.
- This happens because stocks in this setting offer a hedge against a declining risk premium.
- The magnitude of intertemporal hedging demand increases with the slope of the value function.

To Summarize

$$\alpha^{LT}(x) = a_0 + a_1 x \text{ **vs** } \alpha^{ST}(x) = a_1' x \text{ with } a_1 > a_1'$$

Intertemporal CAPM and the Cross-Section of Stocks

- We consider an Epstein-Zin investor that holds the market.
- What are the expected returns on individual assets that make this investor not under- or overweight individual assets?
- The FOC

$$E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} = \gamma cov_t(r_{i,t+1}, r_{m,t+1})$$

$$+ (\gamma - 1) cov_t \left(r_{i,t+1}, (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{m,t+1+j} \right)$$

$$= \gamma cov_t(r_{i,t+1}, N_{CF,t+1} - N_{DR,t+1})$$

$$+ (\gamma - 1) cov_t \left(r_{i,t+1}, N_{DR,t+1} \right)$$

$$= \gamma cov_t(r_{i,t+1}, N_{CF,t+1}) - cov_t \left(r_{i,t+1}, N_{DR,t+1} \right)$$

$$= \gamma cov_t(r_{i,t+1}, N_{CF,t+1}) + cov_t \left(r_{i,t+1}, -N_{DR,t+1} \right)$$

- Now define the following betas with respect to each component

$$\left.\begin{aligned}\beta_{i,CF,t} &= \frac{cov_t(r_{i,t+1}, N_{CF,t+1})}{\sigma^2_{mt})} \\ \beta_{i,DR,t} &= \frac{cov_t(r_{i,t+1}, -N_{DR,t+1})}{\sigma^2_{mt})}\end{aligned}\right. \implies \beta_{imt} = \beta_{i,CF,t} + \beta_{i,DR,t}$$

- Hence, we can write the risk premium on asset i as

$$E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} = \beta_{i,CF,t+1} \cdot \gamma\sigma^2_{mt} + \beta_{i,DR,t+1} \cdot \sigma^2_{mt}$$

- The two β's sum to a market beta but have different prices of risk.

Interpretation of Two-Beta CAPM

$$E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} = \beta_{i,CF,t+1} \cdot \gamma \sigma_{mt}^2 + \beta_{i,DR,t+1} \cdot \sigma_{mt}^2$$

$$\beta_{imt} = \beta_{i,CF,t} + \beta_{i,DR,t}$$

- Shocks to Cash-Flows have a larger price of risk than shocks to discount rates when $\gamma > 1$.
- This is because *long-term investors fear permanent decline in wealth drive by cash-flows than they fear temporary declines in wealth due to higher discount rates.*

Application to Growth Stocks

- Growth stocks have high beta and low returns.
- Two beta model can reconcile this by making the beta of growth stocks to come disproportionately more from Discount Rate beta.
- This means that growth stocks should do particularly well when future returns go down (risk premium goes down) and do particularly bad when future returns go up (risk premium goes up)

Three Beta Model

- Recall that volatility directly affects SDF of the Epstein-Zin investor.
- Campbell et al (2017) shows that SDF can be written as

$$\tilde{m}_{t+1} = -\gamma N_{CF,t+1} + N_{DR,t+1} + \frac{1}{2} N_{RISK,t+1}$$

$$\text{where } N_{RISK,t+1} = (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j Var_{t+j}(m_{t+1+j} + r_{t+1+j})$$

- By making an additional assumption that market returns and conditional variances follows an VAR(1).

$$N_{RISK,t+1} = \omega(E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j \sigma_{m,t+j} = \omega N_{V,t+1}$$

Three Beta Model

- Put all of this together to get

$$E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} = \gamma\sigma_{mt}^2\beta_{CF,t} + \sigma_{mt}^2\beta_{DR,t} - \frac{\omega\sigma_{mt}^2}{2}\beta_{i,V,t}$$

- For stocks that go up when volatility increases, risk premium is low as they provide a hedge against increases in uncertainty.

Empirical Findings

- Find the growth stocks tend to outperform value stocks when long-term volatility forecasts increase (think about the technology boom of late 90s and the global financial crisis).

- Growth stocks have positive variance beta and since the risk of this covariance is negative in equation, they provide a hedge against deteriorating investment opportunities due to higher volatility.

Intertemporal CAPM: Set Up

ICAPM originates in Merton (1973) set up in continuous time. We'll see a discrete time approach that heavily utilizes approximations

- Suppose that the distribution of the vector of returns R returns and income is a function of the vector of state variables X.
- Define the value function to be $V(W, X)$ where W is wealth.
- Denote α to be a vector of portfolio weights.
- On infinite horizon with consumption at each point in time agents maximize

$$\max_{\alpha_t, C_t} E_t \sum_{s=t}^{\infty} \delta^{s-t} u(C_t)$$

  by choosing portfolio allocation α and consumption C.

- Budget constraint is

$$W_{t+1} = Y_{t+1} + (W_t - C_t)\alpha' R_{t+1}.$$

- Bellman equation for this problem is

$$V(W, X) = \max_{\alpha, c} \left\{ u(C) + \delta \mathbb{E}[V(X_{t+1}, Y_{t+1} + (W_t - C_t)\alpha' R_{t+1}) | X_t = X] \right\}.$$

Intertemporal CAPM: Derivation

- The fundamental equation of asset pricing

$$1 = E_t[M_{t+1}R_{i,t+1}] \implies E_t[R_{i,t+1}] = \frac{1}{E_t M_{t+1}} - \frac{1}{E_t M_{t+1}} cov_t(M_{t+1}, R_{i,t+1})$$

- One period SDF for CCAPM $M_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}$ to obtain

$$\mathbb{E}_t[R_{i,t+1}] = \underbrace{\frac{u'(C_t)}{\beta \mathbb{E}_t u'(C_{t+1})}}_{\text{Ignore going forward}} - \frac{1}{\mathbb{E}_t u'(C_{t+1})} cov_t\left(u'(C_{t+1}), R_{i,t+1}\right)$$

- Use the **Envelope Condition** $V_W(X, W) = u'(C)$:

$$\mathbb{E}_t[R_{i,t+1}] = -\frac{1}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} cov_t\left(V_W(X_{t+1}, W_{t+1}), R_{i,t+1}\right)$$

Intertemporal CAPM: Derivation

- Linearize $V_W(X_{t+1}, W_{t+1})$ around $(X_t, W_t - C_t)$

$$\underbrace{V_W(X_{t+1}, W_{t+1})}_{\text{marginal value of wealth}} = \underbrace{V_W(X_t, W_t - C_t)}_{V_{W,t}} + \sum_{i=1}^{K} \underbrace{V_{WX_i}(X_t, W_t - C_t)}_{V_{WX_i,t}} \Delta X_{i,t+1}$$
$$+ \underbrace{V_{WW}(X_t, W_t - C_t)}_{V_{WW,t}}(\Delta W_{t+1} + C_t)$$

- Substitute $V_W(X_{t+1}, W_{t+1})$ into the Euler equation

$$\mathbb{E}_t[R_{i,t+1}] = -\sum_{j=1}^{K} \frac{X_{j,t} V_{WX_j}(X_t, W_t - C_t)}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} cov\left(\frac{\Delta X_{j,t+1}}{X_{jt}}, R_{i,t+1}\right)$$
$$- \frac{(W_t - C_t) V_{WW}(X_t, W_t - C_t)}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} cov_t\left(\frac{\Delta W_{t+1} + C_t}{W_t - C_t}, R_{i,t+1}\right)$$

Intertemporal CAPM: Derivation

- Use the intertemporal budget constraint

$$W_{t+1} = Y_{t+1} + (W_t - C_t)\alpha' R_{t+1} \implies \frac{\Delta W_{t+1} + C_t}{W_t - C_t} = \alpha' R_{t+1} - 1 + \frac{Y_{t+1}}{W_t - C_t}$$

to get the final ICAPM expression

$$\mathbb{E}_t[R_{i,t+1}] = -\sum_{j=1}^{K} \frac{X_{j,t} V_{WX_j}(X_t, W_t - C_t)}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} cov_t\left(\frac{\Delta X_{j,t+1}}{X_{jt}}, R_{i,t+1}\right)$$

$$- \frac{(W_t - C_t) V_{WW}(X_t, W_t - C_t)}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} cov_t\left(\alpha' R_{t+1}, R_{i,t+1}\right)$$

$$- \frac{(W_t - C_t) V_{WW}(X_t, W_t - C_t)}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} cov_t\left(\frac{Y_{t+1}}{W_t - C_t}, R_{i,t+1}\right)$$

- Note that labour income and returns enter the expression "in the same way", so we'll ignore the last term in the next slides.

Intertemporal CAPM: Breaking Down the Prices of Risk

$$\mathbb{E}_t[R_{i,t+1}] \approx -\sum_{j=1}^{K} \frac{X_{j,t} V_{WX_j}(X_t, W_t - C_t)}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} cov_t \left(\frac{\Delta X_{j,t+1}}{X_{jt}}, R_{i,t+1} \right) - \frac{(W_t - C_t) V_{WW}(X_t, W_t - C_t)}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} cov_t \left(\alpha' R_{t+1}, R_{i,t+1} \right)$$

- Let's start with the price of risk on $cov_t \left(\alpha' R_{t+1}, R_{i,t+1} \right)$.
  - Use a 1st order approx., and we'll get a familiar term

$$-\frac{(W_t - C_t) V_{WW}(X_t, W_t - C_t)}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} \approx -\frac{(W_t - C_t) V_{WW}(X_t, W_t - C_t)}{V_W(X_t, W_t - C_t)}$$

  - It is the relative risk aversion of a representative investor's value function!

- With CRRA utility function, value function inherit the same risk aversion coefficient. Hence, with CRRA utility this term is approximately γ.

- Price of risk on the labor covariance term is the same since labor income and returns on wealth enter in a similar way

Intertemporal CAPM: Breaking Down the Prices of Risk

$$\mathbb{E}_t[R_{i,t+1}] \approx -\sum_{j=1}^{K} \frac{X_{j,t} V_{WX_j}(X_t, W_t - C_t)}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} cov\left(\frac{\Delta X_{j,t+1}}{X_{jt}}, R_{i,t+1}\right) + \dots$$

- Now we deal with the state variables. Use the same approximation

$$-\frac{X_{j,t} V_{WX_j}(X_t, W_t - C_t)}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} \approx -\frac{X_{j,t} V_{WX_j}(X_t, W_t - C_t)}{V_W(X_t, W_t - C_t))} = -\frac{\partial \log V_W(X_t, W_t - C_t)}{\partial \log X}$$

negative of the elasticity of marginal value of wealth to the state variable.

Intertemporal CAPM: Breaking Down the Prices of Risk

- Let's consider what does this means in the case of power utility.

	$V(W,X)$	$V_W(W,X)$	$V_{WX}(W,X)$	V_X	V_{WX}/V_X
$\gamma \neq 1$	$f(X)\dfrac{W^{1-\gamma}}{1-\gamma}$	$f(X)W^{-\gamma}$	$W^{-\gamma}\dfrac{\partial f(X)}{\partial X_j}$	$\dfrac{\partial f(X)}{\partial X_j}\dfrac{W^{1-\gamma}}{1-\gamma}$	$\dfrac{1-\gamma}{W}$
$\gamma = 1$	$A\log(W)+g(X)$	AW^{-1}	0	$\dfrac{\partial g(X)}{\partial X}$	0

You can verify the for of the value function by plugging it into the Bellman equation and observing that wealth cancels. In a finite horizon problem $f(X) \rightarrow f_t(X)$ *and* $g(X) \rightarrow g_t(X)$

Intertemporal CAPM: Breaking Down the Prices of Risk

$$\mathbb{E}_t[R_{i,t+1}] \approx -\sum_{j=1}^{K} \frac{X_{j,t} V_{WX_j}(X_t, W_t - C_t)}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} cov\left(\frac{\Delta X_{j,t+1}}{X_{jt}}, R_{i,t+1}\right) + \dots$$

Consider the case of log utility $\gamma = 1 \implies V_{WX} = 0$

- The marginal utility of wealth doesn't depend on state variables. Hence, the price of risk of return-state covariance is zero $\implies$ myopic demand
- Another case of myopic demand considered in class is iid return. In our framework this simply means

$$cov\left(\frac{\Delta X_{j,t+1}}{X_{jt}}, R_{i,t+1}\right) = 0 \ \forall i$$

Intertemporal CAPM: Breaking Down the Prices of Risk

$$\mathbb{E}_t[R_{i,t+1}] \approx -\sum_{j=1}^{K} \frac{X_{j,t} V_{WX_j}(X_t, W_t - C_t)}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} cov\left(\frac{\Delta X_{j,t+1}}{X_{jt}}, R_{i,t+1}\right) + \dots$$

$$\frac{V_{WX}}{V_X} = \frac{1-\gamma}{W}$$

For $\gamma > 1$ state variables matter.

- Suppose that X_j is such that indicates "good times", i.e. high investment opportunities going forward. Using the formulas above this means

$$V_{WX} = \frac{1-\gamma}{W} \underbrace{\frac{\partial V}{\partial X_j}}_{>0} \implies V_{WX} < 0$$

  such state variable lowers the marginal value of wealth.

- This implies

$$-\frac{X_{j,t} V_{WX_j}(X_t, W_t - C_t)}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} > 0$$

  Assets that covary positively with such state variables command a positive risk premium.

- Conversely, for an aggressive investor $\gamma < 1$ marginal value of wealth increases when investment opportunities going forward improve. Therefore,

$$-\frac{X_{j,t} V_{WX_j}(X_t, W_t - C_t)}{\mathbb{E}_t V_W(X_{t+1}, W_{t+1})} < 0$$