ECON 2723, Asset Pricing, Section 2

Jennifer Walsh

September 29, 2023

Topics to cover

- Equilibrium Pricing
  - CAPM
  - Arbitrage Pricing Theory
  - Conditional CAPM
- SDF
  - Stochastic Discount Factor
  - Volatility Bounds

CAPM

CAPM essentially says that the market portfolio is mean-variance efficient: we can't change the variance of a portfolio w/o chaning the mean

- A perturbation of weights of two assets $w_i \to w_i + dw_i$ and $w_j \to w_j + dw_j$ each financed by a change in the weight of the riskless asset such that

$$dE[R_p] = (E[R_i] - R_f)dw_i + (E[R_j] - R_f)dw_j = 0 \Rightarrow dw_j = -\frac{E[R_i] - R_f}{E[R_j] - R_f} dw_i$$

- The variance of this portfolio shouldn't change

$$d\text{var}(R_p) = 2\text{cov}(R_i, R_p)dw_i + 2\text{cov}(R_j, R_p)dw_j = 0$$

$$d\text{var}(R_p) = 2\text{cov}(R_i, R_p)dw_i - 2\text{cov}(R_j, R_p)\frac{E[R_i] - R_f}{E[R_j] - R_f} dw_i = 0$$

- "Set" $j = p$ and cancel terms to get

$$\text{cov}(R_i, R_p) = \text{var}(R_p)\frac{E[R_i] - R_f}{E[R_p] - R_f} \Rightarrow E[R_i] - R_f = \frac{\text{cov}(R_i, R_p)}{\text{var}(R_p)}(E[R_p] - R_f)$$

- If all investors are identical they will demand the same risky portfolio. Since demand = supply, we have $p = m$ and we get the CAPM

Arbitrage Pricing Theory

The main idea of the APT

- In a diversified portfolio, the risk contributed by the idiosyncratic component of security return should be negligible. **Otherwise, we can construct an asymptotic arbitrage**
- The expected return should depend only on the asset's exposure to the common risk sources

Let's formalize this

- Suppose that asset's i excess return is given by market model

$$R_{it}^e = \alpha_i + \beta_{im} R_{im}^e + \varepsilon_{it}, \;\; E_t[\varepsilon_{it}\varepsilon_{jt}] = 0$$

- $E_t[\varepsilon_{it}\varepsilon_{jt}] = 0$ means that components of returns orthogonal to factor R_m are orthogonal between any two assets.
- Consider forming a *well diversified* portfolio of such assets with weights w_i's. Portfolio's return is then

$$R_{pt}^e = \underbrace{\sum_{i=1}^{N} w_i \alpha_i}_{\alpha_p} + \underbrace{\left(\sum_{i=1}^{N} w_i \beta_{im}\right)}_{\beta_{mp}} R_{mt}^e + \underbrace{\sum_{i=1}^{N} w_i \varepsilon_{it}}_{\varepsilon_p}$$

Arbitrage Pricing Theory

$$R^e_{pt} = \underbrace{\sum_{i=1}^{N} w_i \alpha_i}_{\alpha_p} + \underbrace{\left(\sum_{i=1}^{N} w_i \beta_{im} \right)}_{\beta_{mp}} R^e_{mt} + \underbrace{\sum_{i=1}^{N} w_i \varepsilon_{it}}_{\varepsilon_p}$$

$$R^e_{pt} = \alpha_p + \beta_{mp} R^e_{mt} + \varepsilon_{pt}$$

- What is var(ε_p) in terms of the ε_{it}'s?

Arbitrage Pricing Theory

$$R^e_{pt} = \underbrace{\sum_{i=1}^{N} w_i \alpha_i}_{\alpha_p} + \underbrace{\left(\sum_{i=1}^{N} w_i \beta_{im}\right)}_{\beta_{mp}} R^e_{mt} + \underbrace{\sum_{i=1}^{N} w_i \varepsilon_{it}}_{\varepsilon_p}$$

$$R^e_{pt} = \alpha_p + \beta_{mp} R^e_{mt} + \varepsilon_{pt}$$

- What is var(ε_p) in terms of the ε_{it}'s?
- Suppose that $\alpha_p > 0$
- Do the following trade:
  - Go long \$1 worth of this portfolio $\implies$ "get" $\alpha_p + \beta_{pm} R^e_{mt} + \varepsilon_p$
  - Go short \$$\beta_{pm}$ of market $\implies$ "lose" $\beta_{pm} R^e_{mt}$

  This means that our net gain is $\alpha_p + \varepsilon_p$

Arbitrage Pricing Theory

$$R^e_{pt} = \underbrace{\sum_{i=1}^{N} w_i \alpha_i}_{\alpha_p} + \underbrace{\left(\sum_{i=1}^{N} w_i \beta_{im}\right)}_{\beta_{mp}} R^e_{mt} + \underbrace{\sum_{i=1}^{N} w_i \varepsilon_{it}}_{\varepsilon_p}$$

$$R^e_{pt} = \alpha_p + \beta_{mp} R^e_{mt} + \varepsilon_{pt}$$

- What is var(ε_p) in terms of the ε_{it}'s?
- Suppose that $\alpha_p > 0$
- Do the following trade:
  - Go long \$1 worth of this portfolio $\implies$ "get" $\alpha_p + \beta_{pm} R^e_{mt} + \varepsilon_p$
  - Go short \$$\beta_{pm}$ of market $\implies$ "lose" $\beta_{pm} R^e_{mt}$

  This means that our net gain is $\alpha_p + \varepsilon_p$
- What is the Sharpe ratio on our portfolio? In a simple case when $w_i = \frac{1}{N}$ (but in general when each w_i is "not too large") the variance of portfolio residual is

$$\text{var}(\varepsilon_p) = \text{var}\left(\sum_{i=1}^{N} \frac{1}{N} \varepsilon_i\right) = \sum_{i=1}^{N} \frac{1}{N^2} \text{var}(\varepsilon_i) = \frac{\sigma^2}{N} \to 0$$

  since covariances are zero.
- The Sharpe ratio of this trade is

$$S = \frac{E[\alpha_p + \varepsilon_p]}{\sqrt{Var_t(\varepsilon_{pt})}} = \frac{\alpha_p}{\sigma/\sqrt{N}} \to \infty$$

- This is too attractive. Investors will come in to close this **asymptotic arbitrage** and drive α_p to zero.

Arbitrage Pricing Theory

- If our factors are not portfolios, for example leverage of broker dealers as in Adrian, Etula and Muir (2014), we can use them in the following way

$$R_{it} = \mu_i + \sum_{k=1}^{K} \beta_{ik} f_{kt} + \varepsilon_{it} \quad \text{where } E[\varepsilon_i \varepsilon_j] = 0$$

  then the implication of APT is that

$$\alpha_i \equiv \mu_i - \lambda_0 - \sum_{k=1}^{K} \beta_{ik} \lambda_k \approx 0 \implies \mu_i \approx \lambda_0 + \sum_{k=1}^{K} \beta_{ik} \lambda_k$$

  where
  - λ_0 – zero beta rate (risk free rate if it is available)
  - λ_k – price of risk of factor k – implictly defined by cross-sectional regression of mean returns on a constant and loadings $\overline{R}_i^e = \sum_{k=1}^{K} \lambda_k \widehat{\beta}_{ik} + \alpha_i$
- Essentially what we are doing is reducing the dimension of covariance matrix

$$Cov(R_{it}^e, R_{it}^e) = Cov\left(\sum_{k=1}^{K} \beta_{ik} f_{kt} + \varepsilon_{it}, \sum_{k=1}^{K} \beta_{jk} f_{kt} + \varepsilon_{it}\right) = \beta_i' \underbrace{\Sigma(f)}_{\text{Norm to } I} \beta_j + \begin{pmatrix} \sigma_1 & 0 & \cdots \\ 0 & \sigma_2^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

  For N assets and K factors have $NK + N$. Compare to $N(N+1)/2$ in full covariance matrix

Some thoughts

What is the main difference between APT and equilibrium models?

- In equilibrium models we've seen, in principle, we can price any asset (will return to it in Chapter 4). This is an example of absolute pricing, but it is highly model dependent.
- APT gives the price of the asset relative to other assets. But it doesn't specify, for example, what is the return on the market.
  - If we assume that the market is priced correctly, APT allows us to price other securities relative to the market. (And we discover their α's).
  - If we assume that the market model holds, APT asymptotically yields the CAPM formula without requiring the restrictive and unrealistic assumptions of the latter. ($\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \alpha_i^2 = 0$.)

Stochastic Discount Factor, Recap

- Recall from the lecture that M in complete markets is

$$P = \sum_s q(s)X(s) = \sum_s \pi(s)\frac{q(s)}{\pi(s)}X(s) = \sum_s \pi(s)M(s)X(s) = E[MX]$$

- Divide through by P to get fundamental equation in return form

$$\begin{aligned} 1 &= E[M(1+R)] \\ &= E[M]E[1+R] + Cov(M, 1+R) \end{aligned}$$

Same equation for risk free asset gives

$$1 = E[M](1+R^f)$$

subtract the two to get

$$0 = E[M]E[R-R^f] + Cov(M, 1+R)$$

$$\implies E[R-R^f] = -(1+R^f)Cov(M, 1+R)$$

- Utility maximization with initial wealth W_0:

$$\text{Max}\, u(C_0) + \sum_{s=1}^{S} \beta\pi(s)u(C(s)) \quad \text{subject to}\, C_0 + \sum_{s=1}^{S} q(s)C(s) = W_0$$

implies

$$M(s) = \beta\frac{U'(C(s))}{U'(C_0)}$$

Existence of SDF

Law of one price (weaker than absence of arbitrage) + portfolio formation $\Rightarrow$ There exists an SDF that prices all payoffs within a set of tradable payoffs/assets

- Proof by construction: find a linear combination of payoffs/assets that prices all assets.

$$\mathbb{X} = (X_1 \dots X_N)' = \begin{pmatrix} X_1(1) & \cdots & X_N(1) \\ \vdots & \ddots & \vdots \\ X_1(S) & \cdots & X_N(S) \end{pmatrix}' \text{ and } X^* = \mathbb{X}'c^*$$

$$\begin{pmatrix} P(X_1) \\ \vdots \\ P(X_N) \end{pmatrix} = \begin{pmatrix} \mathbb{E}[X_1' X^*] \\ \vdots \\ \mathbb{E}[X_N' X^*] \end{pmatrix}$$

Write the condition in matrix form as

$$\mathbb{P} = \mathbb{E}[\mathbb{X}\mathbb{X}'c^*] \implies c^* = \left(\mathbb{E}[\mathbb{X}\mathbb{X}']\right)^{-1}\mathbb{P} \implies \mathbb{X}'c^* = X^* = \mathbb{X}\left(\mathbb{E}[\mathbb{X}\mathbb{X}']\right)^{-1}\mathbb{P}$$

- **Such SDF X^* is not guaranteed to be positive for each state of the world**

Positive SDF $\Leftrightarrow$ Absence of arbitrage (stronger than *law of one price*)

- Proof in the textbook
- **SDF is not guaranteed to lie in the set of tradable assets**

It is important to understand that only the SDF that is a linear combination of asset payoffs is unique. There may be many other SDFs of the form $M = X^* + \varepsilon$, where $\mathrm{E}[\mathbb{X}\varepsilon] = \mathrm{E}[\mathbb{X}(M - X^*)] = 0$. When a riskfree asset is traded these must all have higher variance than X^*, as discussed further below in the context of volatility bounds on the SDF. Equivalently, X^* is the projection of every SDF onto the space of tradable payoffs. Thus it can be thought of as the portfolio of available assets that best mimics the behavior of every SDF.

SDF Factor Structure

- We can rewrite the excess return equation to get

$$E[R-R^f] = \underbrace{\frac{Cov(M,1+R)}{Var(M)}}_{\text{Quantity of risk } \beta} \underbrace{\left[-\frac{Var(M)}{E[M]}\right]}_{\text{Price of risk } \lambda}$$

- Suppose that for factors f_k that have mean zero and orthogonal to each other

$$M = a - \sum b_k f_k$$

Risk free rate is $1+R^f = 1/E[M] = a$. Excess return is

$$E[R-R^f] = -(1+R^f)Cov(M,1+R) = -\frac{1}{a}Cov(\sum b_k f_k, R)$$
$$= -\frac{1}{a}\sum b_k Cov(f_k,R) = \sum \frac{Cov(f_k,R)}{Var(f_k)}\left(-\frac{b_k}{a}Var(f_k)\right) = \sum \beta_k \cdot \lambda_k$$

- SDF for CAPM takes the same form $M = a - b(R^W - E[R^W])$. To find a and b can price two assets: risk free bond and the wealth portfolio:

$$1 = E[a-b(R^W - E[R^W])](1+R^f) \implies (1+R^f) = \frac{1}{E[a]}$$
$$1 = E[a-b(R^W - E[R^W])(1+R^W)]$$

Factor structure

- What did we learn?
- Factor models (with mean zero factors f) are equivalent to linear models for the discount factor m, that is

$$\mathbb{E}(1+R^i) = \gamma + \lambda' \beta_i \iff m = a + b' f.$$

- $$E[R - R^f] = \underbrace{\frac{Cov(M, 1+R)}{Var(M)}}_{\text{Quantity of risk } \beta} \underbrace{\left[-\frac{Var(M)}{E[M]}\right]}_{\text{Price of risk } \lambda}$$

- Given a and b from the SDF, can determine γ and λ of the factor model. Conversely, if you know γ and λ from the factor model, you can learn a and b of the SDF.

SDF, Volatility Bounds

If we know the risk free rate, bounds are straightforward

- No distributional assumptions

$$
\begin{aligned}
E[R_{i,t+1} - R_{f,t+1}] &= -(1 + R_{f,t+1})\sigma_t(M_{t+1})\sigma_t(R_{i,t+1})corr_t(M_{t+1}, R_{i,t+1} - R_{f,t+1})) \\
&= -\frac{\sigma_t(M_{t+1})\sigma_t(R_{i,t+1} - R_{f,t+1})corr_t(M_{t+1}, R_{i,t+1} - R_{f,t+1}))}{E_t[M_{t+1}]} \\
&\leq \frac{\sigma_t(M_{t+1})\sigma_t(R_{i,t+1} - R_{f,t+1})}{E_t[M_{t+1}]}
\end{aligned}
$$

- Joint log-normality

$$
\begin{aligned}
0 &= \log E_t[M_{t+1}(1 + R_{i,t+1})] \\
&= E_t[\log(M_{t+1}(1 + R_{i,t+1}))] + \frac{1}{2} Var_t(\log(M_{t+1}(1 + R_{i,t+1}))) \\
&= E_t[m_{t+1}] + E_t[r_{i,t+1}] + \frac{1}{2}\sigma_{mt}^2 + \frac{1}{2}\sigma_{it}^2 + \sigma_{imt}
\end{aligned}
$$

For the **riskless asset** this equation becomes

$$
r_{f,t+1} = -E_t[m_{t+1}] - \frac{1}{2}\sigma_{mt}^2
$$

The **log risk premium with Jensen adjustment**

$$
E_t[r_{i,t+1}] - r_{f,t+1} + \frac{1}{2}\sigma_{it}^2 = -\sigma_{imt} = -\sigma_{it}\sigma_{mt}\rho_{imt} \leq \sigma_{it}\sigma_{mt} \Rightarrow \sigma_{mt} \geq \frac{E_t[r_{i,t+1}] - r_{f,t+1} + \frac{1}{2}\sigma_{it}^2}{\sigma_{it}}
$$

Hansen-Jagannathan Bound

But, we might not know the risk free rate, so we do not know $E[M_{t+1}]$. But, you can make an SDF frontier (boundary), then find $M(\bar{M}) = M^*(\bar{M}) + \varepsilon$. Graphs plot $M^*(\bar{M})$.

Consolidating what you know

- This class has a final exam on December 11. Please contact John if this date is going to be a problem for you.
- Exam will consist of math problems and essay questions. Keeping a record of every model throughout the class will be very helpful.
- For example:
  - Author and year name
  - One sentence description of the abstract
  - Key assumptions
  - Key findings