ECON 2723, Asset Pricing, Section 3

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Admin

- Next problem set due on Nov 2, by 11:59pm.
- I am available after class to discuss.
- Not covered today: Models of habit (class on Tuesday 10/24), any slides titled "(skip)", Barro Disaster model, anything about changing variance.

Contents

- Big idea: Assets are priced by agents who would like to consume things.
- Chapter 1-3 (CAPM): $U(W_{t+1})$. We consumed using the wealth we had after returns were realized.
- Now (like in our consumption example for the SDF in Chapter 4, equations 4.7-4.10), investors solve a two period model:

$$\max u\left(C_0\right) + \sum_{s=1}^{S} \pi(s) u(C(s))$$

subject to

$$C_0 + \sum_{s=1}^{S} q(s) C(s) = W_0,$$

The FOC is:

$$\delta \pi(s) u'(C(s)) = q(s) u'\left(C_0\right) \quad \text{for } s = 1 \ldots S$$

And so

$$M(s) \equiv \frac{q(s)}{\pi(s)} = \frac{\delta u'(C(s))}{u'\left(C_0\right)}$$

- This is the "Consumption CAPM" = "CCAPM"

Power Utility CCAPM

- Simplest Consumption CAPM model with CRRA utility has the following SDF

$$M_{t+1} = \delta \frac{u'(C_{t+1})}{u'(C_t)} = \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$$

in logs

$$m_{t+1} = \log(\delta) - \gamma(c_{t+1} - c_t) = \log(\delta) - \gamma \Delta c_{t+1}$$

- Recall the fundamental equation of asset pricing under joint lognormality

$$1 = E_t[M_{t+1}(1 + R_{t+1})] \implies 0 = E_t r_{i,t+1} + E_t m_{t+1} + \frac{1}{2}\sigma_{it}^2 + \frac{1}{2}\sigma_{mt}^2 + \sigma_{imt}$$

- Risk free rate

$$0 = r_{f,t+1} + E_t m_{t+1} + \frac{1}{2}\sigma_{mt}^2$$

$$r_{f,t+1} = \underbrace{-\log(\delta)}_{\text{time preference}} + \underbrace{\gamma E_t \Delta c_{t+1}}_{\text{intertemporal substitution}} - \underbrace{\frac{\gamma^2}{2}\sigma_{ct}^2}_{\text{precautionary savings}}$$

- Risk premium

$$\mathbb{E}_t[r_{i,t+1}] - r_{f,t+1} + \frac{1}{2}\sigma_{it}^2 = -\sigma_{imt} = \gamma cov_t(r_{i,t+1}, \Delta c_{t+1}) \equiv \gamma \sigma_{ict}$$

Three puzzles

$$m_{t+1} = \log(\delta) - \gamma(c_{t+1} - c_t) = \log(\delta) - \gamma \Delta c_{t+1}$$

$$\mathbb{E}_t[r_{i,t+1}] - r_{f,t+1} + \frac{1}{2}\sigma_{it}^2 = -\gamma \sigma_{ict} \quad (1)$$

$$r_{f,t+1} = -\log(\delta) + \gamma E_t \Delta c_{t+1} - \frac{\gamma^2}{2}\sigma_{ct}^2 \quad (2)$$

Power utility has undesirable properties...

- **Equity Premium Puzzle** (ERP): High average return on equities combined with low σ_{ict} implies very large γ (coefficient of relative risk aversion).
- **Risk Free Rate Puzzle** (RFP): Volatility of consumption growth σ_{ct} is low, so...
  - For small values of γ, the intertemporal substitution term dominates.
  - But, when γ is large, the precautionary savings term dominates: investors react very strongly to increases in uncertainty about future consumption growth by increasing savings and driving down risk-free interest rate.
  - When you calibrate (2) to the γ implied by (1) using the observed $r_{f,t+1}$, δ is far too small. Or, for reasonable δ, $r_{f,t+1}$ is larger than the the T-bill rate.
- **Equity Volatility Puzzle**: Stocks are much more volatile than consumption.
  - This is inconsistent with the aggregate stock market being a claim to the aggregate consumption.

Power Utility vs Epstein-Zin Utility

- The main problem with the power utility is that it tightly links the Relative Risk Aversion (RRA) to the Elasticity of Intertemporal Substitution (EIS)...

$$RRA = \frac{1}{EIS}.$$

- If we try to fit the equity premium puzzle with large RRA...
  - The EIS will be very small.
  - Small EIS means that agents are not willing to substitute consumption across periods.
  - Hence, the risk-free rate is very sensitive to the model parameters.

- Epstein-Zin utility abandons the restriction that $RRA = (EIS)^{-1}$, so can fit the ERP without incurring in the RFP.

Epstein-Zin Preferences (A Simple Example)

Let's unpack the logic of **Recursive Utility** in a simple two period example:

$$U_0 = W\big(C_0, \mu(C_1)\big).$$

- Suppose that the agent consumes deterministic C_0 at $t = 0$ and stochastic C_1 at $t = 1$.
  - We can think about deterministic $\overline{C}_1$ that makes the agent indifferent between C_1 and $\overline{C}_1$.
  - This is **Certainty Equivalent** of C_1: $\overline{C}_1 = \mu(C_1)$.
  - $\mu(\cdot)$ encodes the agent's attitude toward **risk**.

- Now we can aggregate agents lifetime consumption into a lifetime value with aggregator $W(\cdot, \cdot)$:

$$U_0 = W(C_0, \overline{C}_1) = W\big(C_0, \mu(C_1)\big).$$

  - $W(\cdot, \cdot)$ deals with deterministic quantities.
  - So it encodes agents preferences toward **intertemporal substitution**.

Epstein-Zin Preferences (General Form)

Epstein Zin utility is defined as $U_t = f\big(C_t, \mu(U_{t+1})\big)$, where

- $f(\cdot)$ is an aggregator function that evaluates tradeoffs between present and future, with a CES functional form:

$$f(x,y) = \left[(1-\delta)x^{1-\frac{1}{\psi}} + \delta y^{1-\frac{1}{\psi}}\right]^{\frac{1}{1-1/\psi}}.$$

- $\mu(\cdot)$ is a certainty-equivalent function, with CRRA functional form:

$$\mu(U_{t+1}) = \left(E_t U_{t+1}^{1-\gamma}\right)^{\frac{1}{1-\gamma}}.$$

- Hence,

$$U_t = \left[(1-\delta) C_t^{\theta(1-\gamma)} + \delta \left(E_t U_{t+1}^{1-\gamma}\right)^{\theta}\right]^{\frac{\theta}{1-\gamma}}, \text{ with } \theta = \frac{1-1/\psi}{1-\gamma}.$$

- δ is the time discount factor, γ is the coefficient of RRA, and ψ is the EIS.

Epstein-Zin SDF (Main Derivation Steps) - (skip)

- To derive SDF, we follow (essentially) the same perturbation argument as with the standard time separable utility.
  - Denote A-D security price from state s^t to state s_{t+1} as $q(s_{t+1})$.
  - Denote the maximized utility (lifetime value) by U_t.
  - **Perturbation argument:** Decreasing consumption by Δ units, decreases utility by

$$\frac{\partial U_t}{\partial c_t(s^t)}\Delta.$$

This leaves us with Δ units of the numeraire (consumption good) that we can spend on buying $\Delta/q(s_{t+1})$ A-D securities for state s_{t+1}. This increases lifetime value by

$$\frac{\Delta}{q(s_{t+1})}\frac{\partial U_t}{\partial c_{t+1}(s_{t+1})}.$$

At the optimum gains must equal losses,

$$\frac{1}{q(s_{t+1})}\frac{\partial U_t}{\partial c_{t+1}(s_{t+1})} = \frac{\partial U_t}{\partial c_t(s^t)} \implies q(s_{t+1}) = \frac{\partial U_t/\partial c_{t+1}(s_{t+1})}{\partial U_t/\partial c_t(s^t)}.$$

  - For time separable utility this is simple since

$$U_t = E_t\sum_{j=0}^{\infty}\delta^j u(c_{t+j}) = u(c_t) + \sum_s \pi(s)\delta u(c_{t+1}(s)) + E_t\sum_{j=2}^{\infty}\delta^j u(c_{t+j})$$

Epstein-Zin SDF (Main Derivation Steps) - (skip)

- For EZ this more cumbersome...

$$q(s_{t+1}) = \frac{\partial U_t / \partial c_{t+1}(s_{t+1})}{\partial U_t / \partial c_t(s^t)}, \text{ and } U_t = f\left(C_t, \mu(U_{t+1})\right).$$

  - Numerator:

$$\begin{aligned}
\frac{\partial U_t}{\partial c(s_{t+1})} &= \frac{\partial f(C_t, \mu(U_{t+1}))}{\partial c(s_{t+1})} \\
&= f_2(C_t, \mu(U_{t+1})) \frac{\partial \mu(U_{t+1})}{\partial U_{t+1}(s^t, s_{t+1})} \frac{\partial U_{t+1}(s^t, s_{t+1})}{\partial c_{t+1}(s^t, s_{t+1})} \\
&= f_2(C_t, \mu(U_{t+1})) \underbrace{\frac{\partial \mu(U_{t+1})}{\partial U_{t+1}(s_{t+1})}}_{\propto \pi(s_{t+1})} f_1(c_{t+1}(s_{t+1}), \mu_t(U_{t+2}))
\end{aligned}$$

  - Denominator is easy

$$\frac{\partial U_t}{\partial C_t(s^t)} = f_1(C_t, \mu(U_{t+1}))$$

  - Then you plug in the function form for f and μ and carefully calculate each of the terms (Look at notes).
  - The SDF is then

$$M_{t+1}(s_{t+1}) = \frac{q(s_{t+1})}{\pi(s_{t+1})}.$$

Epstein-Zin and the Budget Constraint

- Now we want to transform the SDF,

$$M(s^t, s_{t+1}) = \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \left(\frac{U_{t+1}}{\mu_t(U_{t+1})} \right)^{-\left(\gamma - \frac{1}{\psi}\right)},$$

  into something with empirical content that can be tested in the data.

- Continuation utility is not observed, but we can relate it to consumption and wealth.

- Intertemporal budget constraint of an agent who is living off his wealth is (6.42)

$$W_{t+1} = (W_t - C_t)(1 + R_{w,t+1}),$$

  where $R_{w,t+1}$ is the return on the wealth portfolio.

Transforming the SDF (skip)

Step 1: We conjecture that the

$$U_t = U_t(W_t, X_t) = \phi(X_t)W_t = \phi_t W_t,$$

where X_t is a vector of state variables that are correlated with future returns.

- The problem becomes

$$U_t(W_t, X_t) = \max_{C_t, v_t} \left\{(1-\delta)C_t^{\rho} + \delta \left[E_t[\phi_{t+1}^{\alpha}(1+R_{w,t+1})^{\alpha}]\right]^{\rho/\alpha}(W_t - C_t)^{\rho}\right\}^{1/\rho},$$

  where v_t represents a vector of portfolio weights on the different assets available (see next slides).

- Take first order condition with respect to consumption

$$(1-\delta)\rho C_t^{\rho-1} - \delta(\mu_t^*)^{\rho}\rho(W_t - C_t)^{\rho-1} = 0,$$

  where

$$\mu_t^* \equiv \left[E_t[\phi_{t+1}^{\alpha}(1+R_{w,t+1})^{\alpha}]\right]^{1/\alpha}$$

  and

$$\rho = 1 - 1/\psi, \ \alpha = 1-\gamma.$$

- This implies a consumption rule of the form

$$C_t = \eta_t W_t.$$

- When $\psi = 0 \implies \rho = 1 \implies C_t/W_t = const.$

Transforming the SDF (skip)

$$U_t(W_t, X_t) = \left\{(1-\delta)C_t^{\rho} + (\mu_t^*)^{\rho}(W_t - C_t)^{\rho}\right\}^{1/\rho}$$

Step 2: Derive $(\mu_t^*)^{\rho}$ as a function of η using the FOC in the last slide and the conjectured rule for consumption:

$$(\mu_t^*)^{\rho} = \left(\frac{\eta_t}{1-\eta_t}\right)^{\rho-1} \frac{(1-\delta)}{\delta}.$$

Step 3: Plug $(\mu_t^*)^{\rho}$ together with $C_t = \eta_t W_t$ into U_t:

$$\begin{aligned} U_t(W_t, X_t) &= \left[(1-\delta)C_t^{\rho} + \delta(\mu_t^*)^{\rho}(W_t - C_t)^{\rho}\right]^{1/\rho} \\ &= \underbrace{(1-\delta)^{1/\rho}\eta_t^{\frac{\rho-1}{\rho}}}_{\phi_t} W_t = \phi_t W_t \end{aligned}$$

This proves our initial conjecture for the form of the value function.

Transforming the SDF (skip)

Step 4: Rewrite FOC for consumption in terms of observable quantities to arrive at

$$1 = \delta E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{\alpha \frac{\rho - 1}{\rho}} (1 + R_{W,t+1})^{\alpha/\rho} \right]^{\rho/\alpha}$$

This looks like a fundamental pricing equation applied to wealth portfolio.

Step 5: Solve the portfolio choice problem. Since this choice affects only the continuation utility, the maximization problem can be written as

$$\max_{v_t} E_t[U_{t+1}^{\alpha}]^{1/\alpha} \text{ s.to } v_t'\mathbf{1} = 1$$

$$\iff$$

$$\max_{v_t} E_t[\phi_{t+1}^{\alpha} W_{t+1}^{\alpha}]^{1/\alpha} \text{ s.to } v_t'\mathbf{1} = 1$$

$$\iff$$

$$\max_{v_t} E_t[\phi_{t+1}^{\alpha}(W_t - C_t)(1 + R_{W,t+1})^{\alpha}]^{1/\alpha} \text{ s.to } v_t'\mathbf{1} = 1$$

Transforming the SDF (skip) - continuation of step 5

Step 5: FOC w.r.t. $v_t^{(i)}$ implies

$$E_t\left[\left(\phi_{t+1}(1+R_{W,t+1})\right)^{\alpha}\right]^{1/\alpha-1} E_t\left[\phi_{t+1}^{\alpha}(1+R_{W,t+1})^{\alpha-1}(1+R_{i,t+1})\right] = \underbrace{\lambda}_{\text{Lagrange mult.}}$$

- Multiply both sides by $v_t^{(i)}$ and sum FOCs across assets to get

$$E_t[(\phi_{t+1}(1+R_{W,t+1}))^{\alpha}]^{1/\alpha-1} E_t\left[\phi_{t+1}^{\alpha}(1+R_{W,t+1})^{\alpha-1}(1+R_{W,t+1})\right] = \lambda$$

- This simplifies to

$$E_t[(\phi_{t+1}(1+R_{W,t+1}))^{\alpha}]^{1/\alpha} = \lambda$$

- Plug this into FOC from above to get

$$E_t\left[\frac{\phi_{t+1}^{\alpha}(1+R_{W,t+1})^{\alpha-1}}{E_t[(\phi_{t+1}(1+R_{W,t+1}))^{\alpha}]}(1+R_{i,t+1})\right] = 1$$

- **Note that this is the ICAPM**: SDF contains both the return on the wealth portfolio, R_W, and investment opportunities, captured by ϕ_{t+1}.

Transforming the SDF (skip)

Step 6: Finally, plug in ϕ_t in terms of $\eta_t = C_t/W_t$ and do a bunch of algebra to arrive at

$$E_t \left[\delta^{\rho/\alpha} \left(\frac{C_{t+1}}{C_t} \right)^{\alpha \frac{\rho-1}{\rho}} (1 + R_{W,t+1})^{\frac{\alpha}{\rho}-1} (1 + R_{i,t+1}) \right] = 1$$

- The SDF is defined as a random variable M_{t+1} such that $E[M_{t+1} R_{i,t+1}] = 1$ for all i.
- Hence, from the equation above:

$$M_{t+1} = \delta^{\rho/\alpha} \left(\frac{C_{t+1}}{C_t} \right)^{\alpha \frac{\rho-1}{\rho}} (1 + R_{W,t+1})^{\frac{\alpha}{\rho}-1}$$

or rewriting in John's notation

$$M_{t+1} = \left(\delta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \right)^{\theta} \left(\frac{1}{1 + R_{w,t+1}} \right)^{1-\theta}$$

- Assuming joint log-normality the log-SDF in surprise form is

$$\tilde{m}_{t+1} = -\frac{\theta}{\psi} \tilde{c}_{t+1} - (1-\theta) \tilde{r}_{w,t+1}$$

Epstein-Zin Risk-Premium

Eventually we almost get to (6.43) $m_{t+1} = \theta \log(\delta) - \frac{\theta}{\psi} c_{t+1} - (1-\theta) r_{w,t+1}$

- Deriving the risk premium for Epstein-Zin preferences is easy

$$\begin{aligned} E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} &= -cov_t(\tilde{m}_{t+1}, r_{i,t+1}) \\ &= -cov_t\Big(-\theta \frac{\tilde{c}_{t+1}}{\psi} - (1-\theta)\tilde{r}_{w,t+1}, r_{i,t+1}\Big) \\ &= \theta \frac{1}{\psi} \underbrace{\sigma_{ic}}_{\text{CCAPM}} + (1-\theta) \underbrace{\sigma_{iw}}_{\text{"CAPM"}} \end{aligned}$$

- With $\gamma = \frac{1}{\psi}$ and $\theta = 1$ we have the standard consumption CAPM.
  When $\gamma = 1$, $\psi \neq 1$ and $\theta = 0$ we have the static CAPM.

- We want to know what the risk free rate is. But, deriving it is somewhat more challenging.
  - Since the mean SDF contains mean return on wealth portfolio, we *first* need to price the wealth portfolio.

- Apply the above equation to wealth portfolio itself

$$E_t r_{w,t+1} - r_{f,t+1} + \frac{\sigma_w^2}{2} = \frac{\theta}{\psi}\sigma_{wc} + (1-\theta)\sigma_w^2$$

$$E_t r_{w,t+1} = r_{f,t+1} - \frac{\sigma_w^2}{2} + \frac{\theta}{\psi}\sigma_{wc} + (1-\theta)\sigma_w^2$$

Epstein-Zin Risk-Free Rate

- From the previous slide:

$$m_{t+1} = \theta \log(\delta) - \frac{\theta}{\psi} c_{t+1} - (1-\theta) r_{w,t+1}$$

$$E_t r_{w,t+1} = r_{f,t+1} - \frac{\sigma_w^2}{2} + \frac{\theta}{\psi}\sigma_{wc} + (1-\theta)\sigma_w^2$$

- Standard pricing equation for the risk free asset implies

$$0 = E_t m_{t+1} + E_t r_{i,t+1} + \frac{\sigma_m^2}{2} + \frac{\sigma_i^2}{2} + \sigma_{im} \implies$$

$$\begin{aligned} r_{f,t+1} &= -E_t m_{t+1} - \frac{\sigma_m^2}{2} \\ &= -E_t\left[\theta \log(\delta) - \frac{\theta}{\psi} c_{t+1} - (1-\theta) r_{w,t+1}\right] \\ &\quad - \frac{1}{2} Var_t\left(\theta \log(\delta) - \frac{\theta}{\psi} c_{t+1} - (1-\theta) r_{w,t+1}\right) \end{aligned}$$

Epstein-Zin Risk-Free Rate

- Plug in for $E_t r_{w,t+1}$ and solve for $r_{f,t+1}$:

$$r_{f,t+1} = -\log(\delta) + \frac{1}{\psi} E_t \Delta c_{t+1} - \frac{1}{2}\frac{\theta}{\psi^2}\sigma_c^2 + \frac{\theta - 1}{2}\sigma_w^2$$

- Compare to power utility risk-free rate

$$r_{f,t+1} = -\log(\delta) + \gamma E_t \Delta c_{t+1} - \frac{\gamma^2}{2}\sigma_{ct}^2$$

Extended Consumption CAPM (CCAPM+)

- Okay,

$$E_t[r_{i,t+1}] - r_{f,t+1} + \frac{\sigma_i^2}{2} = \theta \frac{\sigma_{ic}}{\psi} + (1-\theta)\sigma_{iw}$$

  But can we say σ_{ic} and σ_{iw} are independently measurable quantities?

- Because the budget constraint restricts the behavior of consumption in relation to the wealth portfolio return, *innovations in consumption* are linked to *innovations in wealth*.

- For the **extended CAPM**, we rewrite the SDF *solely* in terms of **consumption** and **consumption growth**.

- Nests CCAPM and adds aversion to fluctuations in long-run consumption growth.

Extended Consumption CAPM – Wealth Portfolio

- Assuming constant second moments (homoskedasticity) we have, from expressions in previous slides,

$$E_t r_{w,t+1} = const + r_{f,t+1} = const + \frac{1}{\psi} E_t \Delta c_{t+1}$$

$$\Longrightarrow \; E_{t+j} r_{w,t+1+j} = const + \frac{1}{\psi} E_{t+j} \Delta c_{t+1+j}$$

$$\Longrightarrow \; E_{t+1} r_{w,t+1+j} = const + \frac{1}{\psi} E_{t+1} \Delta c_{t+1+j}, \text{ and}$$

$$E_t r_{w,t+1+j} = const + \frac{1}{\psi} E_t \Delta c_{t+1+j}$$

$$\Longrightarrow \; (E_{t+1} - E_t) r_{w,t+1+j} = \frac{1}{\psi} (E_{t+1} - E_t) \Delta c_{t+1+j}$$

Extended Consumption CAPM – Wealth Portfolio

- Use return approximation (in news terms) for the wealth portfolio that pays consumption as dividends and substitute DR part with $\tilde{r}_{w,t+1+j}$ from last slide...

$$
\begin{aligned}
\tilde{r}_{w,t+1} &= (E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j \Delta c_{t+1+j} - (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j r_{w,t+1+j} \\
&= (E_{t+1} - E_t)\Delta c_{t+1} + (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j \Delta c_{t+1+j} - \frac{1}{\psi}(E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j \Delta c_{t+1+j} \\
&= (E_{t+1} - E_t)\Delta c_{t+1} + \left(1 - \frac{1}{\psi}\right)(E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j \Delta c_{t+1+j} \\
&= \tilde{c}_{t+1} + \left(1 - \frac{1}{\psi}\right) \tilde{g}_{t+1}
\end{aligned}
$$

Extended Consumption CAPM – Wealth Portfolio

$$\tilde{r}_{w,t+1} = \underbrace{\tilde{c}_{t+1}}_{\textit{current dividend}} + \underbrace{\left(1 - \frac{1}{\psi}\right)\tilde{g}_{t+1}}_{\textit{innov. future consump. growth}}$$

- Current dividend on wealth portfolio (current consumption) affects the value of wealth one-for-one, all else equal.
- News about future consumption growth have offsetting effects on the value of wealth.
  1. Consumption growth increases dividend growth one-for-one – *cash flow effect*.
  2. Consumption growth increases the discount rate (log-expected return on wealth) in proportion to the $(EIS)^{-1}$ – *discount rate effect*.
- CF dominates (is dominated by) DR effect if $\psi > 1$ ($\psi < 1$).

Extended Consumption CAPM – Stochastic Discount Factor

$$\tilde{r}_{w,t+1} = \tilde{c}_{t+1} + \left(1 - \frac{1}{\psi}\right)\tilde{g}_{t+1}$$

- Use the return on wealth portfolio to plug in for SDF

$$\begin{aligned}
\tilde{m}_{t+1} &= -\frac{\theta}{\psi}\tilde{c}_{t+1} - (1-\theta)\tilde{r}_{w,t+1} \\
&= -\frac{\theta}{\psi}\tilde{c}_{t+1} - (1-\theta)\left(\tilde{c}_{t+1} + \left(1 - \frac{1}{\psi}\right)\tilde{g}_{t+1}\right) \\
&= -\left(1 + \theta\left(\frac{1}{\psi} - 1\right)\right)\tilde{c}_{t+1} - (1-\theta)\left(1 - \frac{1}{\psi}\right)\tilde{g}_{t+1} \\
&= -\gamma\tilde{c}_{t+1} - \left(\gamma - \frac{1}{\psi}\right)\tilde{g}_{t+1}
\end{aligned}$$

Extended Consumption CAPM – Stochastic Discount Factor

$$\tilde{m}_{t+1} = -\gamma \tilde{c}_{t+1} - \left(\gamma - \frac{1}{\psi}\right) \tilde{g}_{t+1}$$

- The effect of innovations on the SDF in absolute value is *increasing* in γ (the RRA):
  1. A risk-averse agent who does not value consumption smoothing ($\psi = \infty$) has a lower marginal utility of current consumption[1] when current consumption is unexpectedly high or when there are good news about future consumption.
  2. The more risk-averse, then the more the agent benefits from good news.

[1] lower marginal utility of wealth today

Extended Consumption CAPM – Stochastic Discount Factor

$$\tilde{m}_{t+1} = -\gamma \tilde{c}_{t+1} - \left(\gamma - \frac{1}{\psi}\right) \tilde{g}_{t+1}$$

- The effect of innovations on the SDF in absolute value is *decreasing* in ψ (the EIS):
  1. $\psi < \infty$ implies a preference for consumption smoothing[2] so that changes in future expected consumption cause the marginal utility from additional consumption today to move in the same direction.
  2. That is, good news about future consumption growth increase the marginal utility of consumption today pushing the agent towards increasing current consumption.
  3. The higher the EIS, the lower is the agents' desire to smooth consumption, weakening the effect on current marginal utility.

[2] There is complementarity between current and future consumption growth.

Extended Consumption CAPM – Stochastic Discount Factor

$$\tilde{m}_{t+1} = -\gamma \tilde{c}_{t+1} - \left(\gamma - \frac{1}{\psi}\right) \tilde{g}_{t+1}$$

- The relative magnitude of γ and $1/\psi$ determines the *net effect of news about future growth* on the agent's marginal utility from current consumption.
  1. With power utility, $\gamma = 1/\psi$: news about future consumption growth do not affect the agent's marginal utility today.
  2. When $\gamma > 1/\psi$: the agent cares more about an **earlier resolution of uncertainty** than about a **smoother consumption path**. So good news about future consumption growth **reduce** the agent's marginal utility today.
  3. When $\gamma < 1/\psi$: the reverse holds... If investment opportunities improve, the agent's marginal utility from current consumption is higher.

Extended Consumption CAPM – Risk Premium

- Using the standard formula for risk premia in homoskedastic and lognormal setting (and combining the expressions in previous slides):

$$E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} = \gamma \sigma_{ic} + \left(\gamma - \frac{1}{\psi}\right) \sigma_{ig}.$$

  - The higher is σ_{ig}, the more the returns of asset i covaries with *news* about future consumption growth.
  - If $\gamma > 1/\psi$, the agent "likes" future consumption growth since it transforms uncertainty into predictable consumption variation.
  - Thus, if σ_{ig} is high, the asset's return tend to be high (low) when there are good (bad) news and, hence, do not hedge the risk of innovations to consumption growth.

Intertemporal CAPM (ICAPM/CAPM+) (skip)

- For the extended CAPM, we solved for the return on the wealth portfolio in terms of consumption and consumption growth.
- For the **intertemporal CAPM**, however, we solve for consumption in terms of **current** and **future expected return on the wealth portfolio**.

Intertemporal CAPM (skip)

$$\tilde{r}_{w,t+1} = \tilde{c}_{t+1} + \left(1 - \frac{1}{\psi}\right)\tilde{g}_{t+1}$$

Now we're going to substitute out consumption

- Use the expression from above to substitute for dividend growth in return decomposition

$$\begin{aligned}
\tilde{r}_{w,t+1} &= \tilde{c}_{t+1} + (E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j \Delta c_{t+1+j} - (E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j r_{w,t+1+j} \\
&= \tilde{c}_{t+1} + \frac{\psi}{\psi - 1}(\tilde{r}_{w,t+1} - \tilde{c}_{t+1}) - (E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j r_{w,t+1+j}
\end{aligned}$$

$$\implies \tilde{c}_{t+1} = \tilde{r}_{w,t+1} + (1-\psi)\underbrace{(E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j r_{w,t+1+j}}_{\tilde{h}_{t+1}}$$

- And plug this into SDF

$$\begin{aligned}
\tilde{m}_{t+1} &= -\frac{\theta}{\psi}\tilde{c}_{t+1} - (1-\theta)\tilde{r}_{w,t+1} \\
&= -\gamma\tilde{r}_{w,t+1} - (\gamma - 1)\tilde{h}_{t+1}
\end{aligned}$$

- And the risk premium is

$$E_t r_{i,t+1} - r_{f,t+1} + \frac{\sigma_i^2}{2} = \gamma\sigma_{iw} + (\gamma - 1)\sigma_{ih}$$

All Epstein-Zin SDFs (skip)

$$\text{Baseline SDF: } \tilde{m}_{t+1} = -\gamma \tilde{c}_{t+1} - (\tilde{u}_{t+1} - \tilde{c}_{t+1})$$

$$\text{Wealth Return SDF: } \tilde{m}_{t+1} = -\theta \frac{\tilde{c}_{t+1}}{\psi} - (1-\theta)\tilde{r}_{w,t+1}$$

$$\text{Extended Consumption CAPM SDF: } \tilde{m}_{t+1} = -\gamma \tilde{c}_{t+1} - \left(\gamma - \frac{1}{\psi}\right) \tilde{g}_{t+1}$$

$$\text{Intertemporal CAPM SDF: } \tilde{m}_{t+1} = -\gamma \tilde{r}_{w,t+1} - (\gamma - 1)\, \tilde{h}_{t+1}$$

Long-Run Risk Model

- Do prices move because of cash flow news or discount rate news?
  - Cochrane (2005) argues that discount rate news explains the majority of asset price movements, not discount rates.
  - Key piece of evidence: high price-dividend ratios forecast low future returns, not high future cash flows.
- Bansal and Yaron offer a cash flow based explanation which has **persistent variation** in both the growth rate and volatility of aggregate consumption.
- In GE, the same shocks affect expected cash flows and dividends.
  - In a consumption-based model in which the aggregate stock market is a claim to aggregate consumption, expected future consumption growth increases both future dividends and real interest rates.
  - These have competing effects - high dividends raise stock prices, high discount rates lower them. If the EIS is 1, then the two effects cancel and expected future consumption has no effect on prices.

Long-Run Risk Model

- The main equation of the **long-run risk model** is the **extended consumption SDF**.
- What is the risk premium on the wealth portfolio?
- Using the equations for $\tilde{r}_{w,t+1}$ and $\tilde{m}_{t+1}$ from previous slides...

$$
\begin{aligned}
rp_w \equiv E_t[r_{w,t+1}] - r_{f,t+1} + \frac{\sigma_i^2}{2} &= \gamma cov_t(\tilde{c}_{t+1} + \left(1 - \frac{1}{\psi}\right)\tilde{g}_{t+1}, \tilde{c}_{t+1}) \\
&+ \left(\gamma - \frac{1}{\psi}\right) cov_t\left(\tilde{c}_{t+1} + \left(1 - \frac{1}{\psi}\right)\tilde{g}_{t+1}, \tilde{g}_{t+1}\right) \\
&= \gamma\sigma_c^2 + \left(\gamma - \frac{1}{\psi}\right)\left(1 - \frac{1}{\psi}\right)\sigma_g^2
\end{aligned}
$$

Long-Run Risk Model

$$rp_w = \overbrace{\gamma\sigma_c^2}^{\textit{small in data}} + \underbrace{\overbrace{\left(\gamma - \frac{1}{\psi}\right)}^{>0}\overbrace{\left(1 - \frac{1}{\psi}\right)}^{>0}\sigma_g^2}_{>>0 \textit{ to rationalize ERP}}$$

- σ_g^2: is the **variance of revisions** in the present discounted value of expected consumption growth.
  - Interpretation: "quantity" of long-run risk.
- $(\gamma - \frac{1}{\psi})$: captures the extent to which agents dislike the late resolution of risk.
- $(1 - \frac{1}{\psi})$: determines whether the wealth portfolio covaries positively or negatively with revisions in expected consumption growth.
  - When $\psi > 1$, CF effect on wealth dominates the DR effect (see slide 23), so wealth covaries positively with expected consumption growth.

Volatility and Risk Premium (skip)

Bansal and Yaron (2006) relies on the effect of volatility in lowering the consumption wealth ratio to pump up the risk premium. We show in a simple case under what conditions does it work:

- Assume we have the following consumption process

$$c_{t+1} = c_t + g + \varepsilon_{t+1} \implies E_t \Delta c_{t+1} + g \text{ where } var(\Delta c_{t+1}) = var(\tilde{c}_{t+1}) = var(\varepsilon_{t+1}) \equiv \sigma^2$$

  this implies $\tilde{g}_{t+1} = 0$ since consumption is a random walk. We, therefore, have

$$\tilde{r}_{w,t+1} = \tilde{c}_{t+1} + \left(1 - \frac{1}{\psi}\right) \tilde{g}_{t+1} = \tilde{c}_{t+1}$$

- We then substitute $\tilde{r}_{w,t+1}$ into risk premium and risk free rate

$$E_t r_{w,t+1} = r_{f,t+1} - \frac{\sigma_c^2}{2} + \frac{\theta}{\psi}\sigma_c^2 + (1-\theta)\sigma_c^2$$

$$r_{f,t+1} = -\log(\delta) + \frac{1}{\psi} E_t \Delta c_{t+1} - \frac{1}{2}\frac{\theta}{\psi^2}\sigma_c^2 + \frac{\theta - 1}{2}\sigma_c^2$$

- Plug $r_{f,t+1}$ into $E_t r_{w,t+1}$

$$E_t r_{w,t+1} = -\log(\delta) + \frac{g}{\psi} - \frac{\sigma^2}{2}(1-\gamma)\left(1 - \frac{1}{\psi}\right)$$

Volatility and Risk-Premium (skip)

$$E_t r_{w,t+1} = -\log(\delta) + \frac{g}{\psi} - \frac{\sigma^2}{2}(1-\gamma)\left(1 - \frac{1}{\psi}\right)$$

- Use expected return on wealth portfolio for the consumption wealth ratio

$$\begin{aligned}
c_t - w_t &= -E_t \sum_{j=0}^{\infty} \rho^j \Delta c_{t+1+j} + E_t \sum_{j=0}^{\infty} \rho^j r_{w,t+1+j} \\
&= -\sum_{j=0}^{\infty} \rho^j g + \sum_{j=0}^{\infty} \rho^j \left[-\log(\delta) + \frac{g}{\psi} - \frac{\sigma^2}{2}(1-\gamma)\left(1-\frac{1}{\psi}\right)\right] \\
&= \frac{g}{1-\rho} - \frac{\log(\delta)}{1-\rho} + \frac{g}{\psi(1-\rho)} - \frac{1}{1-\rho} \times \frac{\sigma^2}{2}(1-\gamma)\left(1-\frac{1}{\psi}\right) \\
\implies w_t - c_t &\propto \frac{\sigma^2}{2}(1-\gamma)\left(1-\frac{1}{\psi}\right)
\end{aligned}$$

- Hence, consumption-to-wealth ratio decreases with volatility when $(1-\gamma)$ and $(1-1/\psi)$ have the opposite sign.
- When we fix g we fix geometric average return. Higher σ then means higher average arithmetic return. When $\gamma > 1 \implies 1-\gamma < 0$ the agent sees a deterioration of investment opportunities. If $\psi > 1 \implies 1 - 1/\psi > 0$, then the agent has strong intertemporal substitution motives and increases his consumption in response to worse investment opportunities.

Testing EIS > 1 (skip)

- The EIS measures the responsiveness of the growth rate of consumption to the real interest rate.
- The homoskedastic Euler equation is:

$$r_{i,t+1} = \mu_i + \left(\frac{1}{\psi}\right)\Delta c_{t+1} + \eta_{i,t+1}. \tag{3}$$

  Alternatively, we can write flip it and write it as:

$$\Delta c_{t+1} = \tau_i + \psi r_{i,t+1} + \zeta_{i,t+1}. \tag{4}$$

- Central idea: consumers plan to change consumption from t to $t+1$ given their expectations of real interest rates.
  - Actual movements in consumption differ from planned movements by an unpredictable random variable, $\zeta_{i,t+1}$, which is available in $t+1$ which was not incorporated into the planning process the year prior.
  - If expectations of the interest rate shift, then there should be a shift in the rate of change of consumption.

Testing EIS > 1 (skip)

Approaches to estimation (Hall 1988):

- Survey data on expected price changes (returns) on the market
- Assume rational expectations, and suppose that the expected returns are given by the expectation plus noise.
  - We can relate the conditional mean of the rate to observed variables known to consumers when they choose consumption.
  - Suppose the mean of the subjective distribution of rates to be a linear combination of observables with known weights:
    $$\bar{r}_{t+1} = \beta x_{t+1} \tag{5}$$
    and β are known in advance.
  - Then, we can instrument for the interest rate on the RHS with past variables which determine the expected real rate.
  - Since the error term is random from the perspective of $t+1$, these variables are uncorrelated with the error term but they do affect expectations of the real rate.