ECON 2723, Asset Pricing, Section 5

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Admin

- Next problem set due on Nov 7.
- I am available after class to discuss.

Contents

Basic Concepts

- Price of a Zero Coupon Bond (ZCB) that pays 1 at time $t + n$ is P_{nt}. Yield-to-Maturity (YTM) is defined as

$$Y_{nt} : P_{nt} = \frac{1}{(1 + Y_{nt})^n} \implies p_{nt} = -n y_{nt} \implies y_{nt} = -\frac{1}{n} p_{nt}$$

- Compared to equities, bonds are not infinitely lived assets so we need to keep track of time to maturity when calculating returns

$$r_{n,t+1} = p_{n-1,t+1} - p_{nt} = -(n-1) y_{n-1,t+1} + n y_{nt} = \underbrace{y_{nt}}_{\text{initial yield}} + \underbrace{(n-1)(y_{nt} - y_{n-1,t+1})}_{\text{change in yield}}$$

- Decompose return as

$$\begin{aligned} r_{n,t+1} &= y_{nt} + (n-1)(y_{nt} - y_{n-1,t+1}) + (n-1) y_{n,t+1} - (n-1) y_{n,t+1} \\ &= y_{nt} + \underbrace{(n-1)(y_{nt} - y_{n,t+1})}_{\Delta \text{ in yield curve}} + \underbrace{(n-1)(y_{n,t+1} - y_{n-1,t+1})}_{\text{"riding the yield curve"}} \end{aligned}$$

- Forward rate constructed with "no arbitrage" argument:

$$\begin{aligned} f_{nt} &= p_{nt} - p_{n+1,t} \\ &= y_{nt} + (n+1)(y_{n+1,t} - y_{nt}) \end{aligned}$$

y_n is the average rate at which we can borrow for n periods, f_n is the marginal rate at which we can extend borrowing for one more period.

Empirical Regularities

$$r_{n,t+1} = y_{nt} + (n-1)(y_{nt} - y_{n-1,t+1}) + (n-1)y_{n,t+1} - (n-1)y_{n,t+1}$$

$$= y_{nt} + \underbrace{(n-1)(y_{nt} - y_{n,t+1})}_{\Delta \text{ in yield curve}} + \underbrace{(n-1)(y_{n,t+1} - y_{n-1,t+1})}_{\text{"riding the yield curve"}}$$

1. $y_{n-1,t+1} - y_{nt}$ is typically negative and is decreasing with in absolute value with maturity (yield curve is generally concave in maturity). However, its effect is multiplied by $(n-1)$ so that small variations in yield can significantly affect the price of long bonds.

2. Constant maturity spread $y_{n,t+1} - y_{nt}$ is declining over time: we have lower interest rates now, so that the whole yield curve moves down.

3. Excess return $r_{n,t+1} - y_{1t}$ is positive and increases with maturity $\implies$ **term premium**

Expectations Hypothesis

Expectations Hypothesis (EH) stated in logs

$$E_t[r_{n,t+1} - y_{1t}] = \mu_n \neq \text{function of time}$$

In Pure EH $\mu_n = 0$. Alternatively, we can write EH in the following form

$$E_t\left[y_{nt} - \frac{1}{n}\sum_{i=0}^{n-1} y_{1,t+i}\right] = \theta_n \neq \text{function of time}$$

- PEH states that the risk premium on long-term bonds is zero. This is similar to Uncovered Interest Rate Parity in international finance.
- In other works: the PEH states that investors price all bonds as though they were risk-neutral.
- EH states that risk premium is non-zero, but doesn't vary over time. More realistic but we know that risk premium varies over time.

Expectations Hypothesis

$$E_t[r_{n,t+1} - y_{1t}] = \mu_n, \quad E_t\left[y_{nt} - \frac{1}{n}\sum_{i=0}^{n-1} y_{1,t+i}\right] = \theta_n$$

Two representations are equivalent. To this first note that cumulative return of holding the bond to maturity is exactly $n \cdot y_{nt}$:

$$r_{n,t+1}+r_{n-1,t+2}+\cdots+r_{1,t+n} = (p_{n-1,t+1}-p_{nt})+\cdots+(\log(1)-p_{1,t+n-1}) = -p_{nt} = n\cdot y_{nt}$$

Substitute this result into the second representation

$$\begin{aligned}
E_t\left[y_{nt} - \frac{1}{n}\sum_{i=0}^{n-1} y_{1,t+i}\right] &= E_t\left[\frac{1}{n}\left(r_{n,t+1} + r_{n-1,t+2} + \cdots + r_{1,t+n}\right) - \frac{1}{n}\sum_{i=0}^{n-1} y_{1,t+i}\right] \\
&= E_t\left[\frac{1}{n}\sum_{i=0}^{n-1} r_{n-i,t+1+i} - \frac{1}{n}\sum_{i=0}^{n-1} y_{1,t+i}\right] \\
&= E_t\left[\frac{1}{n}\left(\sum_{i=0}^{n-1} r_{n-i,t+1+i} - y_{1,t+i}\right)\right] \\
&= E_t\left[\frac{1}{n}\left(\sum_{i=0}^{n-1} \mu_{n-i}\right)\right] \neq \text{function of time}
\end{aligned}$$

Empirical Predictions of EH

$$E_t[r_{n,t+1} - y_{1t}] = \mu_n, \quad E_t\left[y_{nt} - \frac{1}{n}\sum_{i=0}^{n-1} y_{1,t+i}\right] = \theta_n$$

1. Use the first formulation and substitute for return

$$E_t[y_{nt} - (n-1)(y_{n-1,t+1} - y_{nt}) - y_{1t}] = \mu_n$$
$$E_t[s_{nt} - (n-1)(y_{n-1,t+1} - y_{nt})] = \mu_n$$

$$\implies s_{nt} = \mu_n + E_t[(n-1)(y_{n-1,t+1} - y_{nt})]$$

This says that when spread is large, the future yields are expected to rise to **generate capital losses to offset the initial high yield on the bond**. This prediction motivates the following regression

$$(n-1)(y_{n-1,t+1} - y_{nt}) = \alpha + \beta s_{nt} + \varepsilon_{t+1}$$

where $\beta = 1$ under the null and α is unrestricted.

Empirical Predictions of EH

$$E_t[r_{n,t+1} - y_{1t}] = \mu_n, \quad E_t\left[y_{nt} - \frac{1}{n}\sum_{i=0}^{n-1} y_{1,t+i}\right] = \theta_n$$

2. Rearrange the second formulation to get

$$E_t\left[s_{nt} + y_{1t} - \frac{1}{n}\sum_{i=0}^{n-1} y_{1,t+i}\right] = \theta_n$$

$$s_{nt} = \theta_n + E_t\left[\frac{1}{n}y_{1t} + \frac{1}{n}\sum_{i=1}^{n-1}\left(y_{1t} + \sum_{j=1}^{i}\Delta y_{1,t+j}\right) - n\frac{1}{n}y_{1t}\right]$$

$$s_{nt} = \theta_n + E_t\left[\frac{1}{n}\sum_{i=1}^{n-1}(n-i)\Delta y_{1,t+i}\right]$$

When spread is high, the future short term rates are expected to increase. Moreover, short rates are expected to go up more than the long rates, thus, keeping the spread from exploding.
This motivates the following regression

$$\sum_{i=0}^{n-1}\left(1 - \frac{i}{n}\right)\Delta y_{1,t+i} = \alpha + \beta s_{nt} + \varepsilon$$

Empirical Predictions of EH

$$E_t[r_{n,t+1} - y_{1t}] = \mu_n, \quad E_t\left[y_{nt} - \frac{1}{n}\sum_{i=0}^{n-1} y_{1,t+i}\right] = \theta_n$$

3. The last restriction is about forward rates.

$$\begin{aligned} f_{nt} &= p_{nt} - p_{n+1,t} \\ &= -ny_{nt} + (n+1)y_{n+1,t} \\ &= -\left(n\theta_n + E_t\sum_{i=0}^{n-1} y_{1,t+i}\right) + \left((n+1)\theta_{n+1} + E_t\sum_{i=0}^{n} y_{1,t+i}\right) \\ &= [-n\theta_n + (n+1)\theta_{n+1}] + E_t y_{1,t+n} \\ &= \phi_n + E_t y_{1,t+n} \end{aligned}$$

EH says that variation in forward rates comes from the expectations about future short term rates. In pure EH we have $\phi_n = 0$ so that forward rate is exactly equal to the expectation of short rate

Do the Predictions Hold in the Data?

Not surprisingly no:

1. In regression
$$(n-1)(y_{n-1,t+1} - y_{nt}) = \alpha + \beta s_{nt} + \varepsilon_{t+1}$$
where null is $\beta = 1$ find a negative coefficient lager then 1 on absolute value. This means that large spread predicts fall in long yields.

2. In regression
$$\sum_{i=0}^{n-1} \left(1 - \frac{i}{n}\right) \Delta y_{1,t+i} = \alpha + \beta s_{nt} + \varepsilon$$
where null is $\beta = 1$ we tend to get positive coefficients but which are insignificant and small in magnitude compared to the null hypothesis under EH that $\beta = 1$

Affine Term Structure Models

- Consider the price of an n period zero coupon bond

$$P_{nt} = E_t[M_{t+1}P_{n-1,t+1}] = E_t[M_{t+1}\dots M_{t+n}\cdot 1] = E_t[M_{t,t+n}]$$

modelling the term structure of interest rates is modelling the term structure of SDF.

- In this chapter we work with a reduce form SDF that is not derived from a particular utility function. This has the benefit that it rules out arbitrage. As usual we will assume that everything is jointly lognormal so that we can use the usual

$$\begin{aligned} p_{nt} &= E_t[m_{t+1} + p_{n-1,t+1}] + \frac{1}{2}Var_t(m_{t+1} + p_{n-1,t+1}) \\ &= \underbrace{\left[E_t m_{t+1} + \frac{1}{2}Var_t(m_{t+1})\right]}_{-r_{f,t+1}=-y_{1t}=p_{1t}} + E_t p_{n-1,t+1} + \frac{1}{2}Var_t(p_{n-1,t+1}) + cov_t(m_{t+1}, p_{n-1,t+1}) \\ &= p_{1t} + E_t p_{n-1,t+1} + \frac{1}{2}Var_t(p_{n-1,t+1}) + cov_t(m_{t+1}, p_{n-1,t+1}) \end{aligned}$$

This specifies a recursive equation for p_{nt}.

- Affine models ensure that log bond prices are affine in some state variables.

Risk-Neutral Valuation in Affine Models

- In Chapter 4, we defined the stochastic discount factor as a random variable M such that
$$P(X) = \mathbb{E}[MX],$$
where, in complete markets, $M(s) = \frac{q(s)}{\pi(s)}$.

- Remember from part 4.1.2 (Chapter 4) that we can define risk-neutral probabilities $\pi^Q(s)$ such that
$$\pi^Q(s) = (1 + R_f) \cdot \underbrace{q(s)}_{\text{price contingent claim}} = \frac{M(s)}{\mathbb{E}[M]}\pi(s).$$

  - Hence, we can rewrite the asset pricing equation as
$$P(X) = \left(\frac{1}{1+R_f}\right)\sum_{s=1}^{S}\pi^Q(s)X(s) = \left(\frac{1}{1+R_f}\right)\mathbb{E}^Q[X].$$
  - That is: *the price of any asset is risk-neutral expectation, discounted at the riskless interest rate.*

Risk-Neutral Valuation in Affine Models

- Now let's return to Chapter 8. We want to price bonds under the risk-neutral measure.
- Always start with the bond pricing equation

$$P_{nt} = \mathbb{E}_t \left[M_{t+1} P_{n-1,t+1} \right].$$

- By definition, under the risk-neutral measure Q, bonds must satisfy

$$P_{nt} = \frac{1}{1 + R_{f,t+1}} \mathbb{E}_t^Q [P_{n-1,t+1}]$$

Risk-Neutral Valuation in Affine Models

- In affine models where the (n-dimensional) vector of state variables X_t obeys Gaussian, homoskedastic, and linear dynamics as in

  $$\textit{Physical Law of Motion } (\text{P}): X_{t+1} = \mu + \Phi X_t + \Sigma \varepsilon_{t+1},\ \varepsilon_{t+1} \sim \mathcal{N}(0, I_n),$$

  then *there exists a risk-neutral measure* such that

  $$\textit{Risk-Neutral Law of Motion } (\text{Q}): X_{t+1} = \mu^Q + \Phi^Q X_t + \Sigma \varepsilon^Q_{t+1},\ \varepsilon^Q_{t+1} \sim \mathcal{N}(0, I_n).$$

- As we discussed in Chapter 4, distorted beliefs and risk aversion affect the SDF (and, hence, asset prices) in the same way.
  - When agents are risk-averse, their SDF realtively overweights "bad" states of the world (high mg utility/ high state prices):

    $$M(s) = \frac{q(s)}{\pi(s)},$$

  - Intuitively, the risk-neutral law of montion uses a *distorted data generating process*, **overweighting** states of the world in which the investors' **marginal utility is high**.

Risk-Neutral Valuation in Affine Models

- How to find the parameters that define the risk-neutral measure?
  - Conjecture the log-bond prices are affine in the state variables

$$p_{nt} = A_n + B_n X_t,$$

    and so if X_t is Gaussian, so will be log-prices.
  - Use the pricing equations in slide 47 in logs to get

$$\begin{aligned} p_{nt} &= \mathbb{E}_t[m_{t+1} + p_{n-1,t+1}] + \frac{1}{2} var_t[m_{t+1} + p_{n-1,t+1}] \\ &= -y_{1,t} + \underbrace{\mathbb{E}_t^Q[p_{n-1,t+1}]}_{\text{risk-neutral law of motion}} + \frac{1}{2} var_t[p_{n-1,t+1}]. \end{aligned}$$

  - Now, just match coefficients...

Strong Restrictions

Affine term structure models have strong predictions

- Prices are linear in K factors

$$p_{nt} = A_n + B_n' \mathbf{x}_t$$

  Stack K bond prices

$$\mathbf{p}_t = \mathbf{A} + \mathbf{B}\mathbf{x}_t$$

  Can invert this to get the factors

$$\mathbf{x}_t = \mathbf{B}^{-1}(\mathbf{p}_t - \mathbf{A})$$

- Prices reflect all available information that might be relevant to predict future returns
- This is counterfactual since Cochrane and Piazzesi (2005) show that including lagged forward rates improves predictive power.
- A way to deal with this is to introduce a hidden factor, factor that doesn't affect the term structure but it may be relevant to predict the future dynamics of other state variables.
- Essentially, this means that **B** is no longer invertible. Hence, there exist other factors relevant for predicting returns that are not contained in the term structure.

Completely Affine Homoskedastic Model (skip)

- There is a single state variable x_t that evolves according to

$$x_{t+1} = \mu + \phi x_t + \sigma \varepsilon_{t+1}$$

  Log-SDF is

$$m_{t+1} = -x_t - \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 - \frac{\lambda}{\sigma}\varepsilon_{t+1}$$

- Risk free rate is

$$\begin{aligned} p_{1t} &= E_t m_{t+1} + \frac{1}{2} Var_t(m_{t+1}) \\ &= -x_t - \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 + \frac{1}{2} Var_t\left(\frac{\lambda}{\sigma}\varepsilon_{t+1}\right) \\ &= -x_t \end{aligned}$$

- We conjecture that log bond price is affine in x_t:

$$p_{nt} = A_n + B_n x_t$$

  and use the recursive pricing equation on the previous slide to match coefficients.

Completely Affine Homoskedastic Model (Solving for Bond Price) - (skip)

$$A_n + B_n x_t = -x_t + E_t[A_{n-1} + B_{n-1}x_{t+1}] + \frac{1}{2}Var_t(A_{n-1} + B_{n-1}x_{t+1}) + cov_t(A_{n-1} + B_{n-1}x_{t+1}$$

(some lines of algebra)

$$= \underbrace{\left(A_{n-1} + B_{n-1}\mu + \frac{1}{2}B_{n-1}^2\sigma^2 - \lambda B_{n-1}\right)}_{A_n} + \underbrace{\left(B_{n-1}\phi - 1\right)}_{B_n} x_t$$

- Use the initial conditions $A_1 = 0$ and $B_1 = -1$ to solve for B_n

$$B_n = -1 - \phi - \phi^2 - \cdots - \phi^{n-1} = -\frac{1-\phi^n}{1-\phi} < 0$$

and A_n

$$A_n = A_{n-1} + B_{n-1}(\mu - \lambda) + \frac{1}{2}B_{n-1}^2\sigma^2$$

Notice that λ which is the price of ε-risk only shows up modifying the drift of the process, i.e. in $(\mu - \lambda)$. **Bond prices in a model with risk are the same as in the model without risk but with a lower drift $\mu \to \mu - \lambda$.**

Completely Affine Term Structure Model (Risk Premium) - (skip)

$$x_{t+1} = \mu + \phi x_t + \sigma \varepsilon_{t+1}, \quad m_{t+1} = -x_t - \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 - \frac{\lambda}{\sigma}\varepsilon_{t+1}$$

- Can derive risk premium using the usual formula

$$\begin{aligned} r_{n,t+1} - y_{1t} + \frac{1}{2} Var_t(r_{n,t+1}) &= -cov_t(m_{t+1}, p_{n-1,t+1}) \\ &= \lambda B_{n-1} \end{aligned}$$

  - Decrease in ε_{t+1} drive bond prices up (since $B_n < 0$ in $p_{n,t+1} = A_n + B_n x_{t+1}$). At the same time, this shock drive SDF up when $\lambda > 0$. Hence, bond are doing good in bad times (as measure by higher SDF) and, therefore, they are a hedge. Hence, they have a negative risk premium $\lambda B_{n-1} < 0$ since $B_{n-1} < 0$.
  - When $\lambda < 0$ then the reverse happens $\implies$ bonds go down when SDF goes up and risk premium is positive.

In this case the risk premium is constant and, therefore, the expectation hypothesis holds. However, the pure EH doesn't hold since the risk premium is positive.

Completely Affine Heteroskedastic Model (skip)

- The main drawback of the previous model was constant risk premium. In this model we are going to get time varying risk premium and still an affine price of bonds in the state variable.
- We assume the following law of motion

$$x_{t+1} = \mu + \phi x_t + \sigma x_t^{-1/2} \varepsilon_{t+1}$$

and log-SDF

$$m_{t+1} = -x_t + \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 x_t - \frac{\lambda}{\sigma} x_t^{1/2} \varepsilon_{t+1}$$

- We solve for the log bond prices using the same recursion. However, it is now quadratic in B_n

$$B_n = -1 + (\phi - \lambda) B_{n-1} + \frac{B_{n-1}^2}{2}$$

- Risk premium

$$\begin{aligned} r_{n,t+1} - y_{1t} + \frac{1}{2} Var_t(r_{n,t+1}) &= -cov_t(m_{t+1}, p_{n-1,t+1}) \\ &= \lambda B_{n-1} x_t \end{aligned}$$

is time varying and it is determined by the state x_t. Since, x_t is exactly the the risk free rate, risk premium is proportional to the risk free rate. But we are still unhappy with these results

Completely Affine Heteroskedastic Model (Implications) - (skip)

1. This model predicts that low yield spread $s_{nt} = y_{nt} - y_{1t} = y_{nt} - x_t$ i.e. higher x_t implies large risk premium and, therefore, low yield spread forecasts large return. However, the tests of EH suggest that when spread is low $\implies$ future long yields go up $\implies$ bond prices god down
2. Completely affine term structure model with time varying risk premium implies that variance of bond yields is proportional to risk premium

$$var_t(y_{n,t+1}) = var_t(p_{n,t+1}) = var_t(B_n x_{t+1}) = var_t(B_n \sigma x_t^{1/2}) = B_n \sigma x_t$$

   This doesn't fit the data well. Data from 60s to 80s suggests that volatility increases with a higher power than a square root. On the other hand, more recent data suggests that with lower rates and still high volatility the relationship has a lower power.

Essentially Affine Term Structure Models (skip)

- Duffee (2002) shows that it is possible to write an affine model in which variance varies indepedently of risk premium. In this way we don get get linearity of mean and variance of log SDF in the state variable, but bonds prices and yields are still linear in state.
- We have a homoskedsastic law of motion for the state variable

$$x_{t+1} = \mu + \phi x_t + \sigma \varepsilon_{t+1}$$

but log SDF becomes

$$m_{t+1} = -x_t - \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 x_t^2 - \left(\frac{\lambda}{\sigma}\right) x_t \varepsilon_{t+1}$$

Analagously, the short rate is still x_t

Bond Pricing and Dynamics of Consumption Growth (skip)

- Since interest rates are negatively related to expectations about future SDF the terms

$$cov_t(m_{t+1}, E_{t+1}m_{t+2}) \text{ and } cov_t(m_{t+1}, y_{1,t+1})$$

  have opposite signs.

- **Mean-reverting SDF**

$$cov_t(m_{t+1}, E_{t+1}m_{t+2}) < 0 \implies cov_t(m_{t+1}, y_{1,t+1}) > 0$$

  Bad ($\tilde{m}_{t+1} > 0$) shock to current SDF $\implies$ interest rates go up $\implies$ bond prices go down $\implies$ bonds are risky and command a positive risk premium.

- **Persistent SDF**

$$cov_t(m_{t+1}, E_{t+1}m_{t+2}) > 0 \implies cov_t(m_{t+1}, y_{1,t+1}) < 0$$

  Bad ($\tilde{m}_{t+1} > 0$) shock to current SDF $\implies$ interest rates go down $\implies$ bond prices go up $\implies$ bonds are hedges and command a negative risk premium.

Bond Risk Premia in Homoskedastic Power Utility Model (skip)

- Risk free rate in the power utility model is

$$r_{f,t+1} = -\log(\delta) + \gamma E_t[\Delta c_{t+1}] - \frac{\gamma^2}{2}\sigma_c^2$$

  so that the risk free rate takes the form $r_{f,t+1} = A + Bx_t$ – expected consumption growth is the only state variable.

- The SDF is

$$m_{t+1} = \log(\delta) - \gamma \Delta c_{t+1}$$

- Now suppose there is a process for $E_t[\Delta c_{t+1}]$:

$$E_t[\Delta c_{t+1}] = \mu + \phi E_{t-1}[\Delta c_t] + \sigma \varepsilon_{t+1}$$

  and suppose that the realized consumption is driven by the same shock ε_{t+1}

$$\Delta c_{t+1} = E_t[\Delta c_{t+1}] + \lambda \varepsilon_{t+1}$$

  The SDF then becomes

$$m_{t+1} = \log(\delta) - \gamma E_t[\Delta c_{t+1}] - \gamma \lambda \varepsilon_{t+1}$$

- How does the term premium depends on λ?
  1. When $\lambda > 0$ consumption growth is persistent $\Longrightarrow$ bonds do well in bad times and do good in bad times as discussed earlier $\Longrightarrow$ negative risk premium (hedges)
  2. When $\lambda < 0$ consumption growth is mean-reverting $\Longrightarrow$ bonds perform good in good times and perform badly in bad times as discussed earlier $\Longrightarrow$ positive risk premium