all assets in the payoff space) and, moreover, it is strictly positive, since a solution to the maximization problem above cannot involve a negative gross return in any state. Finally, note that, even though the return to the growth-optimal portfolio is, by construction, in the space of payoffs $\underline{X}$, its reciprocal need not be in $\underline{X}$ when markets are incomplete.

Partial Credit: [2 pts] a solution to the log-utility investor's maximization problem cannot involve negative gross return in any state, showing positivity of the SDF; to ensure log-utility investor's maximization problem, we need to assume: absence of arbitrage [1 pt], and that there exists at least one asset with a gross return that is strictly positive in every state [2 pts].

2. Consider a square-root single-factor affine term structure model, a discrete-time version of the model due to Cox, Ingersoll, and Ross (1985). The model is described by two equations:

$$x_{t+1} = \mu + \phi x_t + \sigma x_t^{1/2} \varepsilon_{t+1},$$
$$m_{t+1} = -x_t - \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 x_t - \left(\frac{\lambda}{\sigma}\right) x_t^{1/2} \epsilon_{t+1},$$

where m_{t+1} is the log stochastic discount factor. The shock ϵ_{t+1} is distributed $N(0,1)$.

a) Show that x_t equals the single-period log real interest rate.

Solution

$$\begin{aligned} y_1 &= -E_t[m_{t+1}] - \frac{1}{2}\mathrm{Var}(m_{t+1}) \\ &= x_t + \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 x_t - \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 x_t \\ &= x_t \end{aligned}$$

b) Write down an equation relating the log price of an n-period real bond to the first and second moments of the log price of an $(n-1)$-period real bond and the log stochastic discount factor.

Solution

Using the conjecture from the next part that prices are linear in x_t (that is, the fact that the model is affine), which is conditionally lognormal and jointly so with the SDF, we have

$$
\begin{aligned}
P_{nt} &= E_t[M_{t+1}P_{n-1,t+1}] \\
p_{nt} &= E_t[m_{t+1}] + E_t[p_{n-1,t+1}] + \frac{1}{2}\text{Var}_t(m_{t+1} + p_{n-1,t+1}) \\
&= E_t[m_{t+1}] + E_t[p_{n-1,t+1}] + \frac{1}{2}\text{Var}_t(m_{t+1}) + \frac{1}{2}\text{Var}_t(p_{n-1,t+1}) + \text{Cov}_t(m_{t+1}, p_{n-1,t+1})
\end{aligned}
$$

Note: Full credit given for second and/or third equation.

c) Conjecture that the price of an n-period real bond can be written as

$$p_{nt} = -ny_{nt} = A_n + B_n x_t.$$

Substitute your conjecture into the equation you derived in part b), and derive equations that can be solved recursively for A_n and B_n.

Solution

$$
\begin{aligned}
p_{nt} &= A_n + B_n x_t \\
&= -x_t + A_{n-1} + B_{n-1}(\mu + \phi x_t) + \frac{\sigma^2}{2}B_{n-1}^2 x_t - B_{n-1}\sigma x^{1/2}\left(\frac{\lambda}{\sigma}\right)x_t^{1/2} \\
&= A_{n-1} + B_{n-1}\mu + x_t\left(-1 + (\phi - \lambda)B_{n-1} + \frac{\sigma^2}{2}B_{n-1}^2\right)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
A_n &= A_{n-1} + B_{n-1}\mu \\
B_n &= -1 + (\phi - \lambda)B_{n-1} + \frac{\sigma^2}{2}B_{n-1}^2
\end{aligned}
$$

Partial Credit: [2 pts] use of the method of undetermined coefficients; [1 pt] algebra with one or more mistakes

d) In this model, the term premium on an n-period bond is proportional to the state variable x_t. Given this fact, explain intuitively (without algebra) why the following two statements are true:

Note: Each subpart is worth 2 points.

(i) Term premia cannot switch sign.

Solution

The sign of the term premium can only switch if the sign of x_t switches. In the continuous-time limit of an Ito process, it is impossible for x_t to switch sign. This is because the time path of x_t is continuous and, if it ever reaches 0, the stochastic term will be zero and x_t will be pushed back to positive territory given that its drift at 0 is $\mu > 0$.

[In the discrete-time version of the CIR model studied here, this is in fact not formally true, as x_t can jump from positive to negative territory and vice versa given a big enough shock (provided we write the instantaneous volatility as $\sigma\sqrt{|x_t|}$, so it is well-defined for negative x_t).]

Note: Full credit given provided answer states that x_t is always positive (regardless of whether it mentions continuous-time limit). No credit otherwise.

(ii) If term premia are positive, then they are inversely related to the yield spread.

Solution

Term premia are increasing in the short rate x_t. The yield spread, $y_{nt} - x_t$, is decreasing in the short rate x_t in this model. An increase in x_t has two opposing forces on the yield spread: first, it raises the term premium, pushing the yield spread up; second, expected future short rates are now lower relative to the current short rate (because of the mean reversion of the stationary process) which means that, for given risk premia, long yields rise by less than the short rate, pushing down the yield spread. The second effect dominates in this model (for realistic parameter values), so that the yield spread is inversely related to x_t and thus inversely related to term premia.

Note: Full credit given if the answer states that the reason yield spreads move inversely with the short rate is the stationarity of the short rate process, so that expectations of future short rates move less than one-for-one with innovations in current short rate. No credit otherwise.

e) An essentially affine single-factor model has a homoskedastic interest rate but a heteroskedastic stochastic discount factor. Write down the model and explain how its properties differ from the square-root model discussed above.

Solution

$$x_{t+1} = \mu + \phi x_t + \sigma \varepsilon_{t+1}$$
$$m_{t+1} = -x_t - \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 x_t^2 - \left(\frac{\lambda}{\sigma}\right) x_t \varepsilon_{t+1}$$

Now the SDF is no longer affine in the state variable, although bond prices remain linear. The interest rate in now homoskedastic. The volatility adjustment is now accounted for by the intercept, A_n of the price function rather than the slope, B_n, since interest rate volatility is constant. The term premium is again proportional in x_t. Since interest rate volatility no longer depends on x_t, the Sharpe ratio is now proportional to x_t rather than to the square root of x_t.

Regarding the two properties from part d), the term premium can now switch sign if x_t switches sign. It also suffers from the counterfactual prediction that term premia are inversely related to the yield spread. However, in this model it is easy to add additional factors to fit time-varying risk premia without affecting interest rate volatility.

Partial Credit: [1 pt] correct statement of laws of motion; [1 pt] term premia can now switch sign (property from d.i.); [1 pt] this model also suffers from prediction d.ii; [1 pt] one other point of comparison between the models.

3. Consider a two-period economy with risky assets whose returns are normally distributed, and a riskfree asset with gross simple return R_f. There are no trading frictions. A risk-averse agent with wealth W_0 in period 0 wishes to maximize his period-1 wealth, $W_1 = (R_f + \gamma R_P)W_0$, where R_P is the excess return to his portfolio of risky assets (here, $1 - \gamma$ is the fraction of the riskfree asset in the agent's overall portfolio).

a) Explain informally why the excess return to any asset i must satisfy

$$E[R_i] = \beta_i E[R_P]$$

where β_i is a constant and R_P is the excess return to the agent's optimal portfolio of risky assets.

Solution

Since returns are normally distributed and the agent is risk-averse, he acts as a mean-variance optimizer. Thus, his portfolio portfolio must be mean-variance efficient (his risky portfolio is the tangency portfolio). Any mean-variance efficient portfolio can serve as the single factor in an expected return-beta model that prices all assets.

Partial Credit: The agent's optimal portfolio is mean-variance efficient [2 pts], because he acts as a mean-variance optimizer, given that returns are normally distributed [1 pt]. Any mean-variance efficient portfolio can serve as the single factor that prices all assets [1 pt].

b) Now suppose the investor is in the process of constructing his optimal portfolio and has a candidate ("benchmark") portfolio of risky assets with excess return R_B that