Econ 70, Personal Finance:

Making Better Decisions and Building a Better Financial System

Section Slides:

Logarithms and Interest

Exponents

- Exponents are used to denote repeated multiplication. For example,

$$\begin{aligned} 2^3 &= 2 \times 2 \times 2 = 8. \\ 10^4 &= 10 \times 10 \times 10 \times 10 = 10,000. \end{aligned}$$

- It follows that **adding** exponents corresponds to **multiplying** the underlying numbers:

$$\begin{aligned} 2 \times 4 &= 2 \times (2 \times 2) = 2^1 \times 2^2 \\ &= 2^{(1+2)} = 2^3 = 8. \\ 1000 \times 10 &= (10 \times 10 \times 10) \times 10 = 10^3 \times 10^1 \\ &= 10^{(3+1)} = 10^4 = 10,000. \end{aligned}$$

Logarithms: A Code Based on Exponents

- The idea of logarithms (logs for short) is to encode a number by the corresponding exponent on a given base.
  - The answer will depend on what base is used.
- For example, if the base is 2, then because $2^3 = 8$, we have $3 = \log_2(8)$. "3 is the exponent on 2 that gives 8, so 3 is the log of 8 in base 2."
- If the base is 10, then because $10^4 = 10,000$, we have $4 = \log_{10}(10,000)$. "4 is the exponent on 10 that gives 10,000, so 4 is the log of 10,000 in base 10."

But Why Do We Bother with Logs Today?

- Today, we don't need logs to do multiplication, division, or exponentiation.
- But they are still extremely useful to represent proportional changes in economic quantities (the level of prices, the value of a portfolio, etc.)
- Compound interest is repeated multiplication, so by thinking in log terms we can understand it as adding up over time.
- Economic variables are often more comprehensible when plotted on a log scale.

Power of Zero and Log of One

- Any number raised to a power zero equals one.
  - $2^0 = 1$, $10^0 = 1$, etc.
- Hence, for any base, the log of one equals zero.

$$
\begin{aligned}
\log_2(1) &= 0 \\
\log_{10}(1) &= 0 \\
\log_b(1) &= 0
\end{aligned}
$$

Log of a Number Close to One

- If the base makes no difference to the log of one, it does make a difference to the logs of any other number.
- Let's look at numbers close to one.

$$
\begin{aligned}
\log_2(1.01) &= 0.01436 \\
\log_3(1.01) &= 0.00906 \\
\log_{10}(1.01) &= 0.00432 \\
\log_e(1.01) &= 0.00995 \approx 0.01.
\end{aligned}
$$

- There is a number $e \approx 2.71828$ (Euler's number) such that the log to base e for a number close to one, say 1.01, is almost exactly that number less one, say 0.01.
- This is a very useful property and so logs with this base are called **natural logarithms** and sometimes written ln instead of log.
- In economics and finance, whenever we work with logs we use natural logs.

Summary of Log Properties

- Multiplication turns into addition

$$\log(A \cdot B) = \log(A) + \log(B)$$

- Division turns into subtraction

$$\log\left(\frac{A}{B}\right) = \log(A) - \log(B)$$

- Exponentiation is easy with logs

$$A = \log_B(B^A)$$

- Logs of 1 always equal zero, logs near 1 almost equal zero

$$\log(1) = 0$$

Practice Problems on Logarithms

1. Find the value of y, when $\log_y 32 = 5$

2. True or False: $\log(a-b) = \log(a) - \log(b)$

3. Simplify the following expression: $\log x^2 y$

Solutions to Practice Problems on Logarithms

1. Find the value of y, when $\log_y 32 = 5$

Ans. $y = 2$

2. True or False: $\log(a-b) = \log a - \log b$

Ans. False, $\log a - \log b = \log(a/b)$

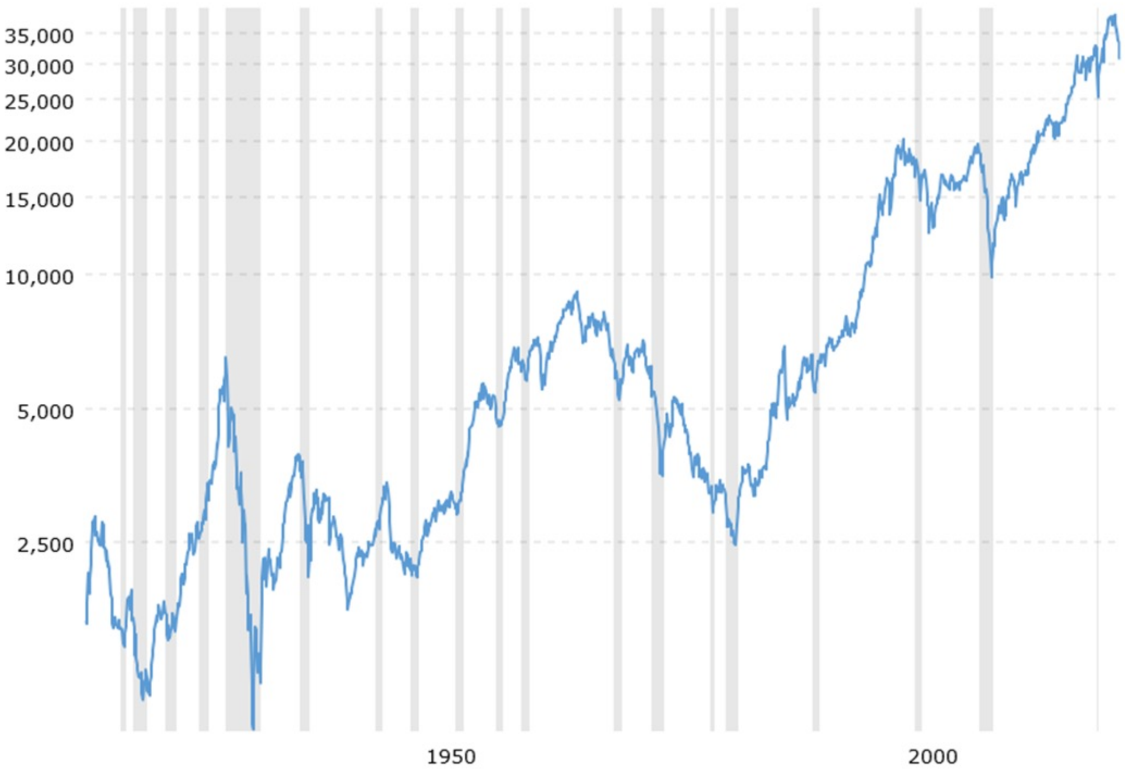
3. Simplify the following expression: $\log x^2 y$

Ans. $2\log x + \log y$

Log Scale

- A log scale in a figure has evenly spaced tick marks that are a constant multiple of one another, say 1, 10, 100, 1,000, etc.
- A series plotted with a log scale shows you the proportional movements in the series.
- Often these are more meaningful in economics than the absolute movements.
- Example:
  - The Dow Jones Index fell 508 points on October 19, 1987. The day before it had closed at 2,247 so this was a 22.6% decline, the largest one-day percentage decline ever measured.
  - The Dow Jones Index fell 2,997 points on March 16, 2020. The day before it had closed at 23,186 so this was a 12.9% decline – much smaller in percentage terms even though much larger in absolute terms.

Dow Jones History: Levels vs Log Scale



Source

Interest

Comparing Monetary Values Over Time

- A basic problem in finance is comparing the values of payments to be made or received at different dates.
- To handle this, we must take account of the interest that you receive if you invest or pay if you borrow.

Interest Over One Period (1)

- Following Kapoor et al., we will write PV for **present value** and FV for **future value**.
- Suppose you put an amount PV in a savings account for one year.
- The **interest rate** is i.
- What do you have in one year?

$$FV = PV(1 + i).$$

- Example: $PV = \$100$, $i = 2\% = 0.02$.

$$\$102 = \$100 \times 1.02.$$

- What you get back in one year is \$100 **principal** (the amount you originally invested) and \$2 **interest**.

Interest Over One Period (2)

- We can rearrange the expression on the previous slide to put the interest rate on the left hand side:

$$1 + i = \frac{FV}{PV}.$$

- Subtract one from each side:

$$i = \frac{FV}{PV} - 1 = \frac{FV - PV}{PV}.$$

- Example: $PV = \$100$, $FV = \$102$.

$$i = \frac{\$102 - \$100}{\$100} = 0.02 = 2\%.$$

Working in Logs (1)

- We have
$$1 + i = \frac{FV}{PV}.$$
- To understand interest, it can be helpful to use **logarithms** (logs):
$$\log(1+i) = \log\left(\frac{FV}{PV}\right) = \log(FV) - \log(PV).$$
- The **natural log** (which has base e, and is sometimes written as ln) has the property that
$$\log(1+x) \approx x$$
when x is small.
- In this course, and in economics generally, we always mean natural logs whenever we refer to logs.

Working in Logs (2)

$$\log(1+i) = \log\left(\frac{FV}{PV}\right) = \log(FV) - \log(PV).$$

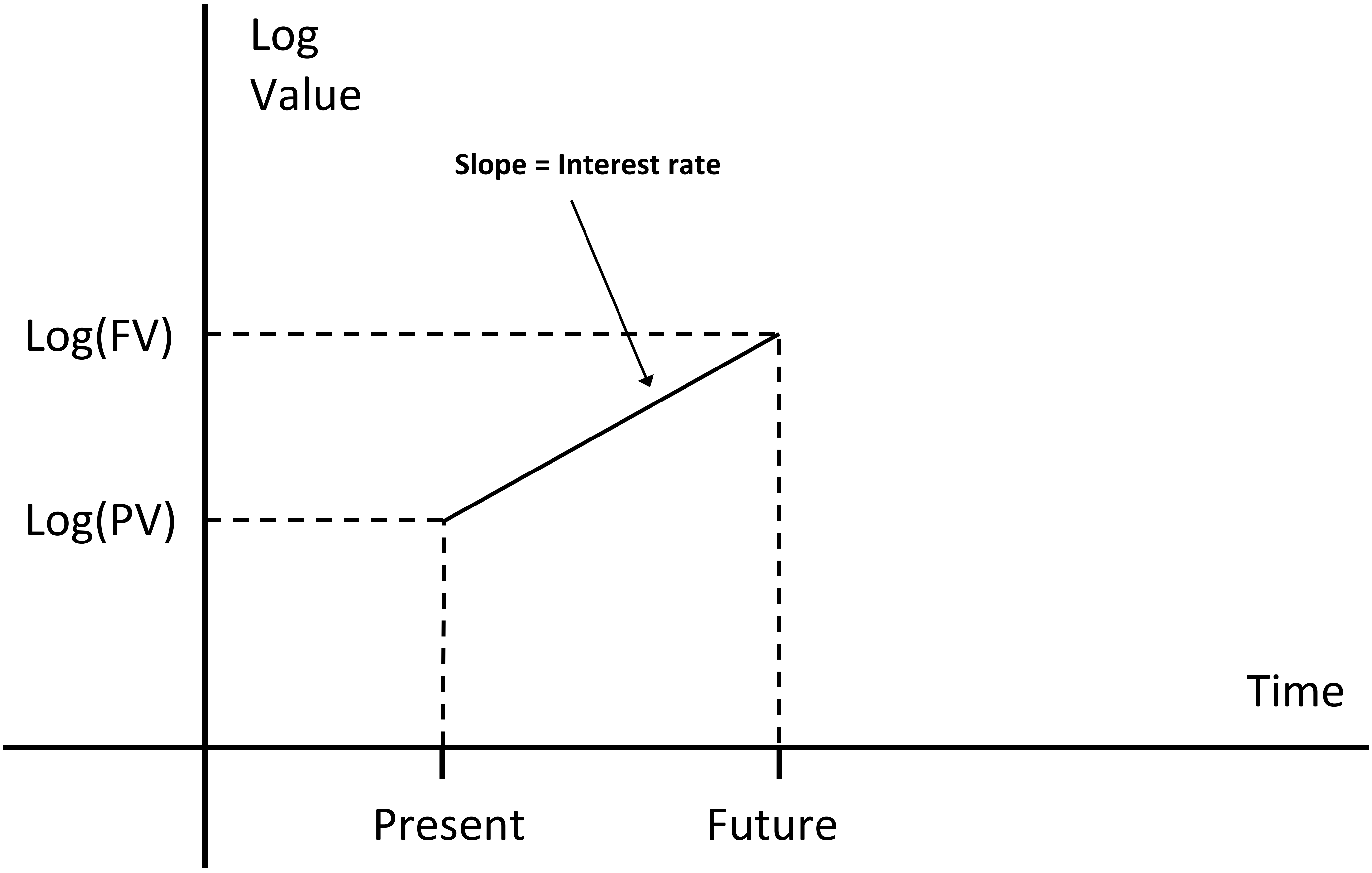
- Since interest rates are small numbers and we are working with natural logs, we have

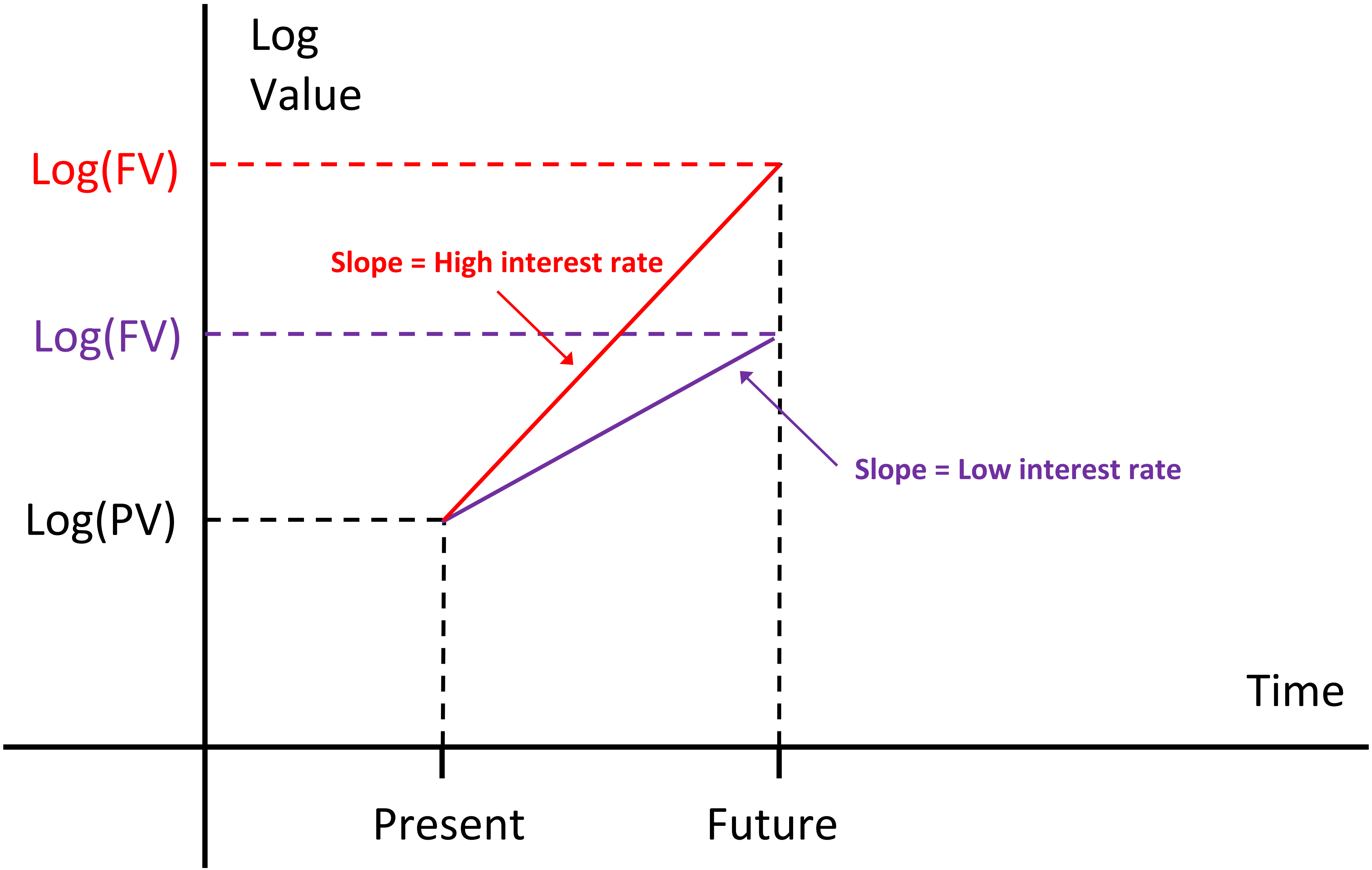
$$i \approx \log(FV) - \log(PV).$$

- Finally, we can use the notation g_V to indicate the growth rate of value $\log(FV) - \log(PV)$. We have

$$i \approx g_V.$$

- The interest rate is just the growth rate of invested value.


Log
Value
Slope = Interest rate
Log(FV)
Log(PV)
Time
Present
Future


Log
Value
Log(FV)
Slope = High interest rate
Log(FV)
Slope = Low interest rate
Log(PV)
Time
Present
Future

Interest Over Many Periods (1)

- Suppose you put an amount PV in a savings account for n years.
- The interest rate is i.
- After one year you have $PV(1+i)$, and the second year you earn interest on that larger amount, not just on your original investment PV. We say that interest **compounds**. Therefore after two years you have $PV(1+i)(1+i) = PV(1+i)^2$.
- What do you have in n years? Write this as $FV(n)$ to indicate that it depends on the number of years you are saving.

$$FV(n) = PV(1+i)^n.$$

- Example: $PV = \$100$, $i = 2\% = 0.02$, $n = 5$. $(1.02)^5 = 1.1041$.

$$FV(n) = \$100 \times 1.1041 = \$110.41$$

Interest Over Many Periods (2)

- What do you have in n years?

$$FV(n) = PV(1+i)^n.$$

- Example: $PV = \$100$, $i = 2\% = 0.02$, $n = 5$. $(1.02)^5 = 1.1041$.

$$FV(n) = \$100 \times 1.1041 = \$110.41.$$

- This is more than \$102 because interest is paid every year. *(Financial literacy test, question 1.)*
- It's also more than \$110 because of compounding. The difference of \$0.41 is small in this example, but large over long periods of time or with higher interest rates.
- What you get back in five years is \$100 principal and \$10.41 interest. If you invest for a longer period of time, you earn more total interest.

Working in Logs Over Many Periods

- What do you have in n years?

$$FV(n) = PV(1+i)^n.$$

$$\frac{FV(n)}{PV} = (1+i)^n.$$

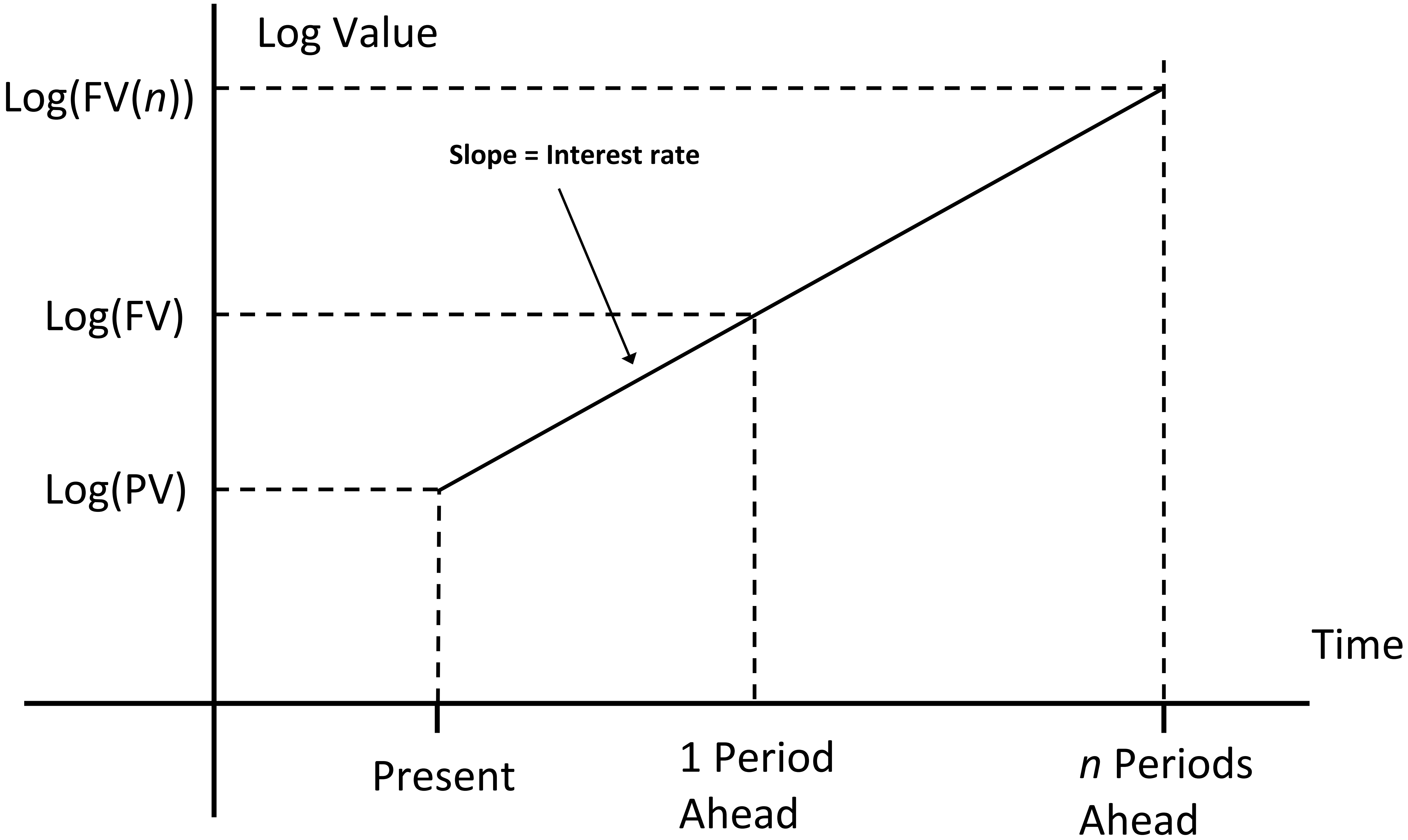
- In logs,

$$\log(FV(n)) - \log(PV) = n\log(1+i) \approx ni$$

or

$$i = \frac{\log(FV(n)) - \log(PV)}{n}$$

- Just as before, the interest rate is the growth rate of value: now an average growth rate over n periods.


Log Value
Log(FV(n))
Slope = Interest rate
Log(FV)
Log(PV)
Time
Present
1 Period Ahead
n Periods Ahead

The Rule of 72

- If you invest at a rate of i, how long does it take to double your money?
- For $PV = 1$, we need to find n such that $FV(n) = 2$.

$$2 = 1(1 + i)^n.$$

- Take logs, remembering that $\log(1) = 0$:

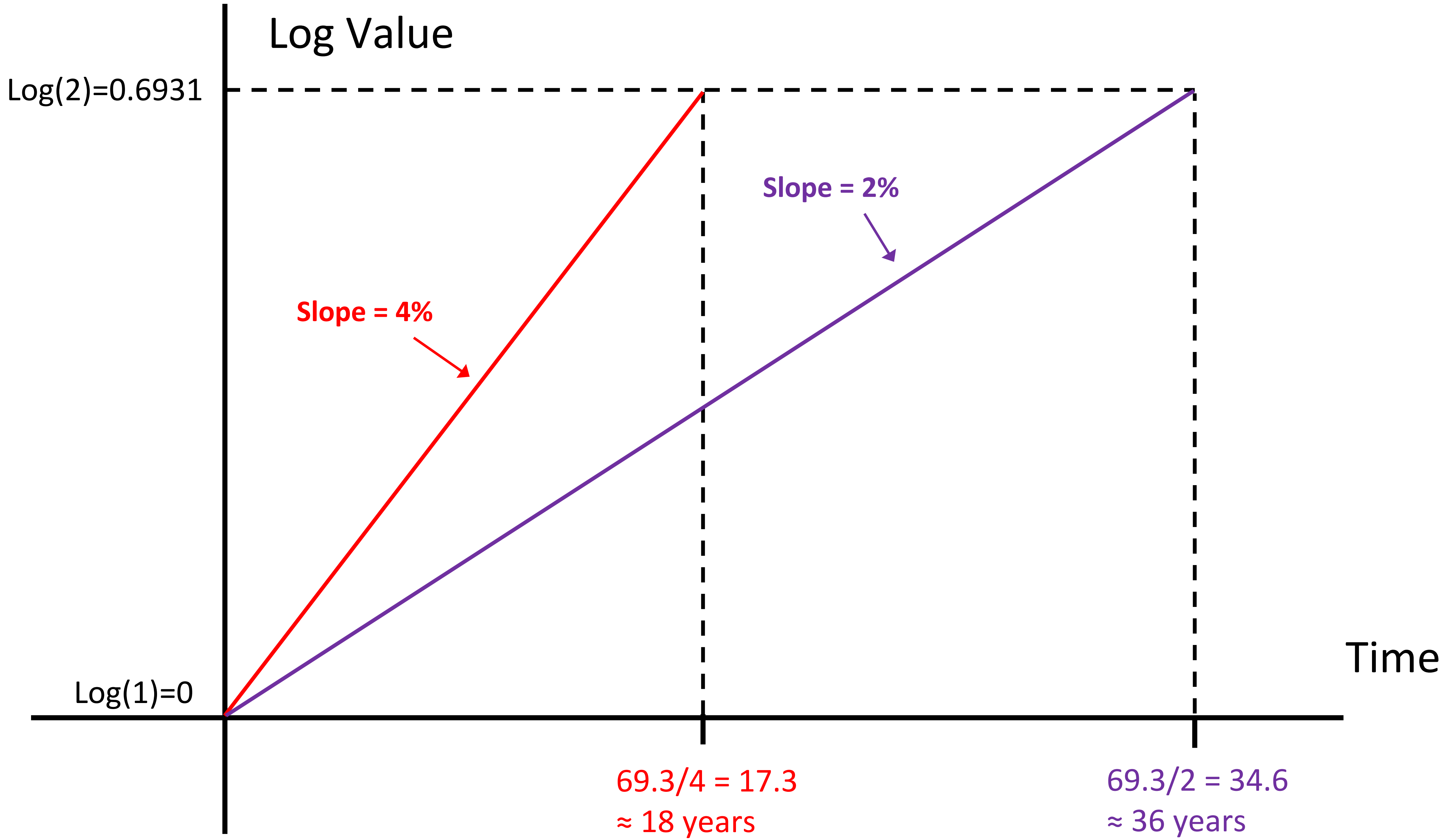
$$\log(2) - \log(1) = \log(2) \approx ni.$$

$$n \approx \frac{\log(2)}{i} = \frac{0.6931}{i} = \frac{69.31}{(i \text{ in } \%)}.$$

- The rule of 72 is the approximation

$$n = \frac{72}{(i \text{ in } \%)}.$$

- So why the rule of 72 and not the rule of 69.31?


Log Value
Log(2)=0.6931
Slope = 2%
Slope = 4%
Time
Log(1)=0
69.3/4 = 17.3
≈ 18 years
69.3/2 = 34.6
≈ 36 years

APPENDIX

A Little Etymology and History

- The word “logarithm” comes from the Greek “logos” meaning proportion, ratio, or word, and “arithmos” meaning number.
  - A logarithm is a “ratio-number”.

- Logarithms were discovered by John Napier
  - A late 16th Century Scot, laird of Merchiston and nicknamed “Marvellous Merchiston” for his mathematical innovations.

How Did People Use Logs?

- Before computers were invented, multiplication was time-consuming and division even more so.
- If you had a table of logs (for any base), you could tackle a multiplication problem like this:

$$
\begin{aligned}
2 \times 4 &= ? \\
\log_2(2) &= 1 \\
\log_2(4) &= 2 \\
1 + 2 &= 3 \\
3 &= \log_2(8) \\
\text{Answer} &= 8
\end{aligned}
$$

- The second and third steps require looking up the logs in the table, and the fifth step requires reverse looking up a number from its log.
- The table of logs is big, but much smaller than a table of all possible number pairs that would be needed to look up the answer directly.

Multiplication and Division

- You can multiply numbers by adding their logs.
- Similarly, you can divide numbers by subtracting their logs.

$$
\begin{aligned}
16 \div 4 &= ? \\
\log_2(16) &= 4 \\
\log_2(4) &= 2 \\
4 - 2 &= 2 \\
2 &= \log_2(4) \\
\text{Answer} &= 4
\end{aligned}
$$

- You can use the same table of logs as before.

Exponentiation

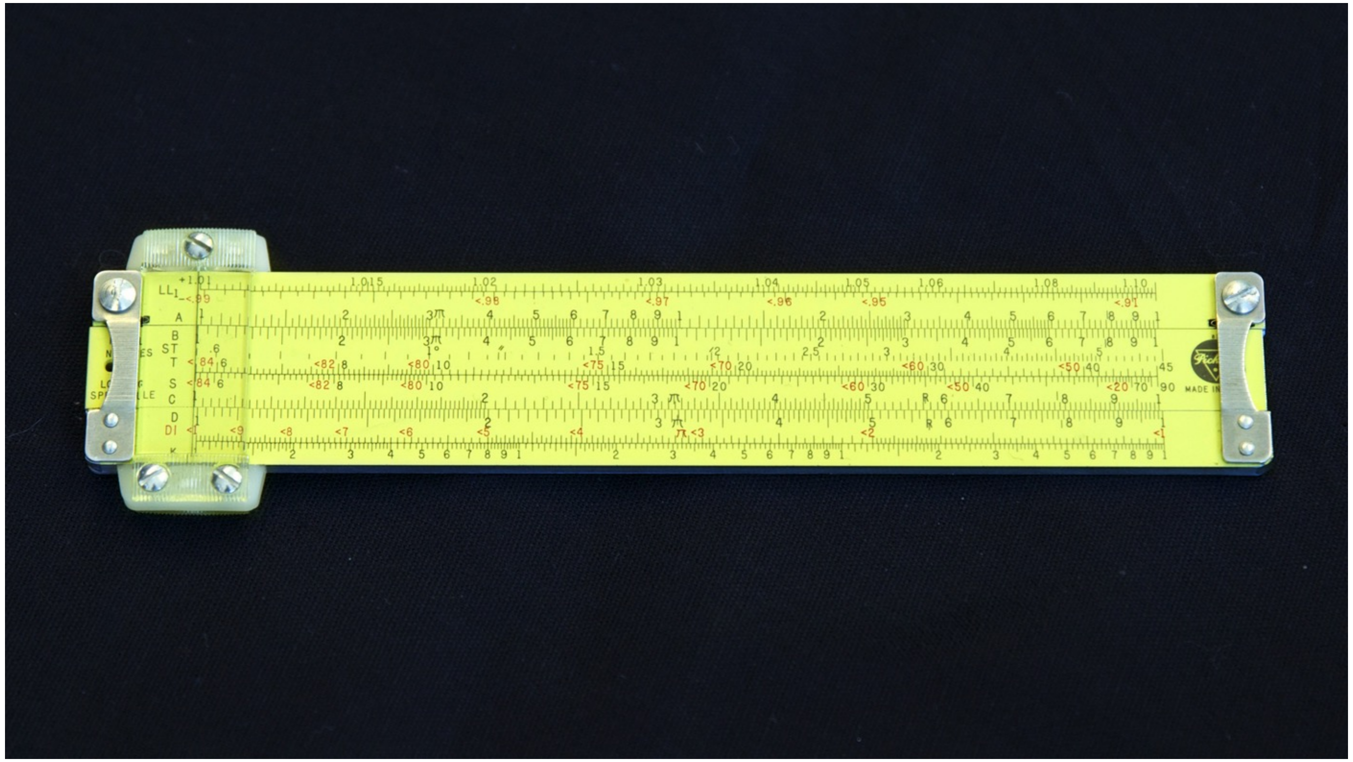
- Exponentiation means raising a number to a power.
  - If the power is 3, this means "multiply the number by itself 3 times".
- You can do this by multiplying the log of the number.

$$
\begin{aligned}
4^3 &= ? \\
\log_2(4) &= 2 \\
3 \times 2 &= 6 \\
6 &= \log_2(64) \\
\text{Answer} &= 64
\end{aligned}
$$

- You can use the same table of logs as before.

The Slide Rule: Log Scales for Computation

- A slide rule has two or more log scales.
- By moving two log scales against one another, one can add or subtract logs and hence multiply or divide.
- The slide rule was widely used for computation through the 1960s.



[Background on the slide rule]