# Math 242: Variations on Cellular Automata

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## 1 Introduction

A cellular automaton consists of a grid of cells, where each cell has two defined quantities: *state* and *neighborhood*. These states are picked from a set of possible states; for a two-state discrete system, the possible states are *on* and *off*. The neighborhood is some set of cells at specific positions relative to a certain cell. Consider a Moore neighborhood ($r = 1$), shown below:

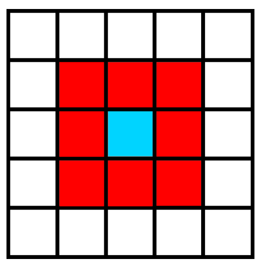


Figure 1: The red cells are the Moore neighborhood of the blue cell.

Based on its current state, the sum of the states of cells in its neighborhood, and a set of update rules, a cell will either remain in the same state or switch to a new one. This update is applied to every cell on the grid to effectively evolve the grid to the next generation.

However, this may still seem hard to visualize. To give a concrete example, John von Neumann, one of the founders of the field, created a self-replicating automaton that he termed the *Universal Constructor*.[1] This self-replication is one of the many features that demonstrates the strength of cellular automata as a tool.

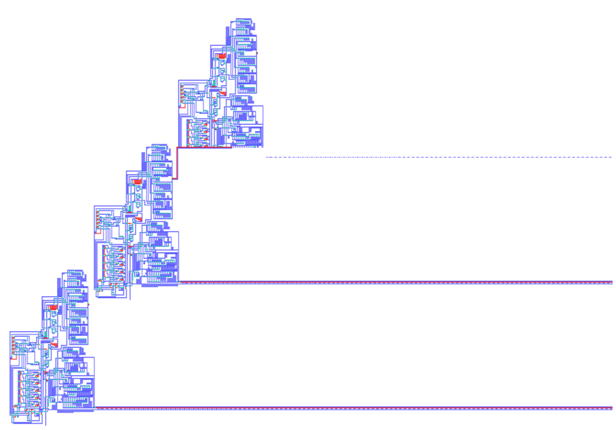


Figure 2: Von Neumann's Universal Constructor

[1] Von Neumann, John. *Theory of Self-Reproducing Automata*. Published by *University of Illinois Press* (Urbana: 1966).

Famously, this mathematical and computational concept first developed in the 1940s became more widely popular as the basis to create simulation games. Specifically, John Conway developed the Game of Life, which was popularized by Martin Gardner in 1970.[2] Conway's general rules for developing the game were threefold:

1. No initial state such that the population can be easily shown to grow infinitely
2. The existence of initial states that seem to grow infinitely
3. Simple initial patterns that stabilize to one of three states: 0 cells alive, a constant configuration, or oscillating configurations

Conway saw that these conditions were sufficient for a cellular automaton that looked organic somehow. See an interesting 3-generation progression of Conway's Game below:

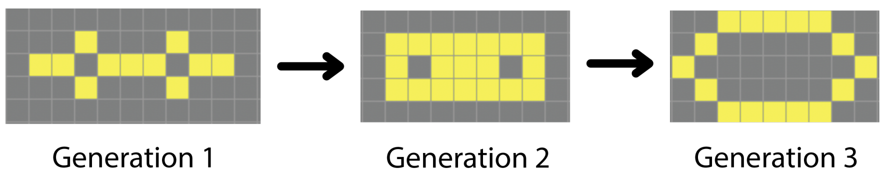



Figure 3: Evolution of Conway's Game for 3 Generations

Now, how does this relate to class and why is it particularly interesting? The theory of cellular automata is the basis of the spatial games proposed in Chapter 9 of Martin Nowak's *Evolutionary Dynamics.*[3] In this framework, states for each cell are actually strategies. Consider the two-state cooperator/defector game. A defector with a size 8 neighborhood of cooperators will get $8b$; a cooperator with 4 cooperators and 4 defectors in its neighborhood would get $4(b-c)+4(-c)$. These sums, representing payoff rather than number of individuals, can then be used to calculate if individuals should switch states or not. Cellular automata are not just simple models of population growth, but can be used to visualize the conditions under which certain evolutionary strategies are viable.

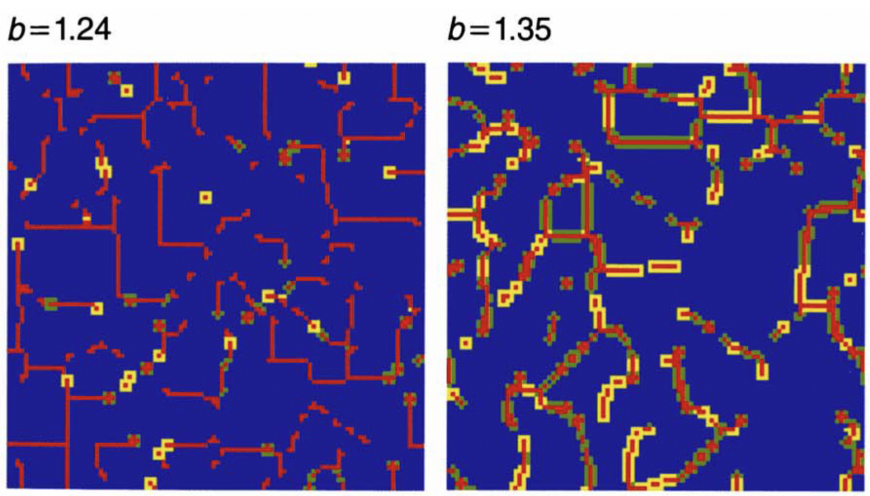



Figure 4: Spatial Games with Varying Reward Values. Blue/Green=Cooperator; Red/Yellow=Defector

The applications to biological modeling with cellular automata are truly broad, ranging from evolution and

[2]Gardner, Martin. "The Fantastic Combinations of John Conway's New Solitaire Game 'Life'." *Scientific American* 223 (Oct 1970): 120-123.

[3]Nowak, Martin. *Evolutionary Dynamics: Exploring the Equations of Life.* Published by *Harvard University Press* (Sep 2006).

ecology to molecular and cellular behavior.[4] With this basic conceptual understanding of the breadth of cellular automata, we can look at the specifics of different systems and how recent researchers have modified them.

## 2 Models

Looking more in detail at Conway's Game of Life, we have the following:

- *States:* On (Live), Off (Dead)
- *Neighborhood:* $r = 1$ Moore Neighborhood, as shown in Figure 1
- *Update Rules:*
  - Survival: A live cell with 2 or 3 live neighbors survives
  - Death: A live cell with 4 or more neighbors dies (overpopulation); a live cell with 0 or 1 neighbors dies (isolation)
  - Birth: A dead cell with 3 live neighbors switches to the live state

While scientists continue to discover new features of Conway's Game,[5] it has been quite thoroughly studied and is generally well understood. Simple variations on the number of states, type of neighborhood, and update rules tend to lead to interesting changes, but these have also been relatively well studied. One especially exciting alteration is to use a continuous set of states, as shown in the image below.

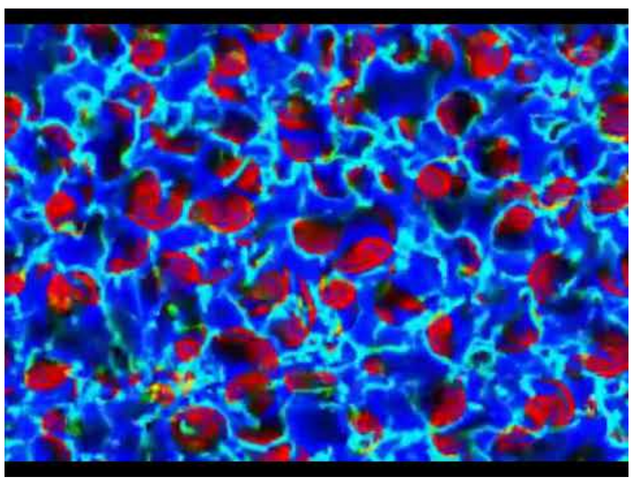


Figure 5: A Continuous-State Cellular Automaton

A researcher known as Slackermanz has thrown an interesting wrench into this concept of grid-based cellular automata, creating Multiple-Neighborhood Cellular Automata (MNCA) in 2014.[6] Instead of just considering a single neighborhood for each cell, multiple neighborhoods are processed instead. Basically, each rule is iterated for the different neighborhoods defined for the cell. This allows for a complex combination of the patterns that are normally generated by one neighborhood. For example, if an $r = 5$ Moore neighborhood creates some small texture and an $r = 35$ Moore neighborhood leads to big structures, applying both neighborhoods during the grid update will create a result that combines the two. See the figures below for an example of the neighborhoods used and the systems generated:

[4] Ermentrout GB and Edelstein-Keshet L. "Cellular Automata Approaches to Biological Modeling." *Journal of Theoretical Biology* 160 (1993): 97-133.

[5] Rendell, Paul. "A Fully Universal Turing Machine in Conway's Game of Life." *Journal of Cellular Automata* 8, no. 1 (2013): 19-38.

[6] Slackermanz. "Understanding Multiple Neighborhood Cellular Automata." May 23, 2021.

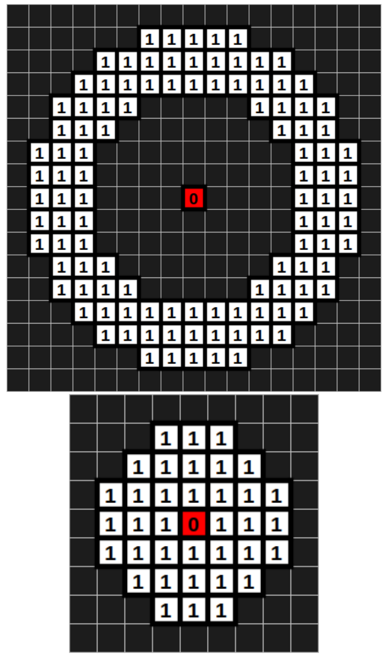


Figure 6: Two Example Neighborhoods Used in MNCA

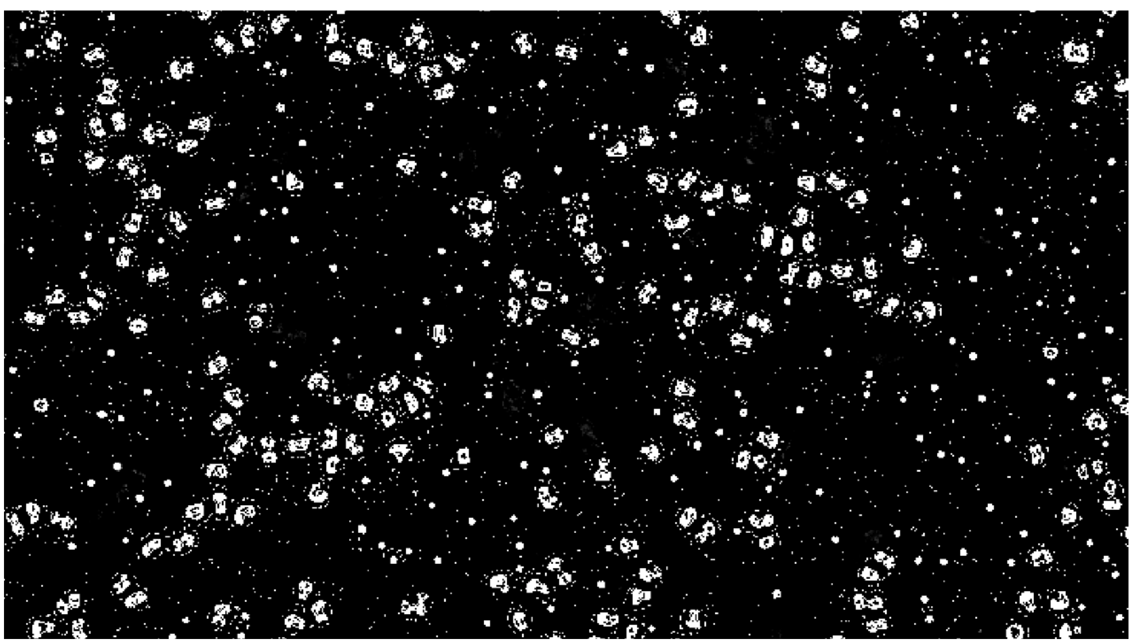


Figure 7: An Image From an Automaton Generated by the Figure 6 Neighborhoods

Some of the most interesting features of MNCA are the emergent properties that show up. These are described in depth in Slackermanz's article,[6] but a few are included here:

- Response to attractive and repulsive forces
- Self-replication
- Interactions between structures that preserve the identities of the original structures
- Collective movement

For my own work, I propose a broad range of variations on cellular automata in the theme of MNCA, which redefines one of the fundamental quantities involved in cellular automata (neighborhoods). While these may be conceptually difficult to interpret, they are computationally easy to implement.

## 2.1 Multiple-State Spatial Games

Instead of "alive/dead" cells that are traditionally considered in Conway's Game, we can consider cell states as describing different strategies, creating a spatial game as discussed in the Introduction. However, rather than restricting the scope of possible strategies to just cooperators and defectors, we can include many more possible states into the game. These could even be stochastic with random errors, simulating the "trembling hand/foggy mind" conditions seen in class. The structures formed in the grid would likely be very different from the simple cooperate-defect game using just a Moore neighborhood.

## 2.2 Evolving Multiple-Neighborhood Cellular Automata

Instead of setting the update rules of some system to be constant over time, we can let them vary. Thus, as the system evolves, so too does the direction of the system, in a sense. I have seen this successfully applied to other organic simulations and presume it will be interesting with cellular automata.

## 2.3 Mutations in Continuous-State Automata

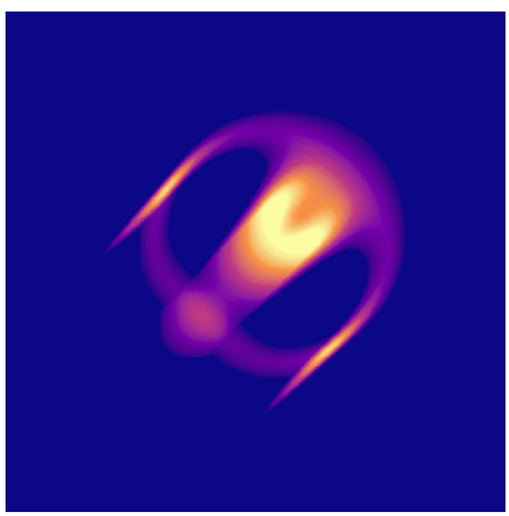
For continuous-state automata ($0 < state < 1$), we can apply mutations during the update process. This would prevent absorbing boundaries from dominating the system and likely create unique interactions about the equilibrium.

## 2.4 Differentiated Cells

Normally, every cell on the grid has the same set of possible states and relative neighborhoods. However, we could try defining certain cells on the grid to have unique neighborhoods. This would differentiate certain positions from others and could be analogous to something like population/environment structure.

## 2.5 Smooth Spatial Games

Instead of the rectangular grid traditionally used in cellular automata, certain researchers have made the system continuous. This was first done in 2011 with SmoothLife,[7] and more recently implemented in Lenia[8] (shown below).



[7]Rafler, Stephan. "Generalization of Conway's 'Game of Life' to a Continuous Domain - SmoothLife." *arXiv* (Dec 2011).
[8]Chan BWC. "Lenia: Biology of Artificial Life." *Complex Systems* 28, no 3 (2019): 251-286.

Figure 8: An Organism from Lenia

These smooth automata look very interesting and could be adapted for use in spatial games.

# 3 Analysis

First, I want to spend time getting comfortable generating cellular automata. I've already messed around with initial conditions for Conway's Game quite a bit, and have begun to code my own version of the game. My general plan for implementation goes as follows: Conway's Game of Life, Cooperator-Defector Spatial Game, Life Games with Multiple Intervals for Update Rules, Multiple-Neighborhood Cellular Automata. I've been working with some code written in C# for Unity from Sebastian Lague's Github to get started. His work is on continuous-state multiple-neighborhood cellular automata, which will take some time to understand. I've included a couple images below.

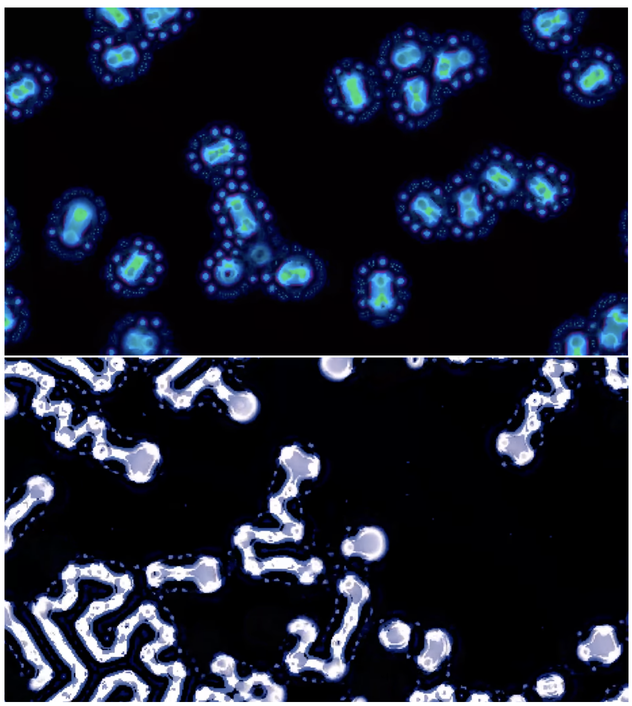


Figure 9: MNCA Images from Sebastian Lague's Video on Complex Behavior

Second, once I'm comfortable generating my own cellular automata with a wide range of specifications, I want to look at the structure of the automata overall. For a given ruleset, are there repeating structures as the system evolves overtime? Do certain neighborhood structures consistently lead to interesting/uninteresting patterns? Can different rulesets lead to similar evolution of the system over time given certain neighborhoods? These questions and many more will take a decent amount of math to answer, which will be guided by intuition developed through experimentation. This categorization will likely be the bulk of my work.

Specifically, I'll be using *local structure theory* for my analysis, which was developed by Howard Gutowitz in 1987.[9] This tool allows one to calculate the invariant quantities for a given cellular automaton and generate a sort of "statistical portrait" to describe the evolution of the system over time. Hopefully, I can generalize this theory to unique automata and classify a broad range of types. I will be looking for repeated structures, stability, and "interesting" automata.

[9] Gutowitz HA, Victor JD, and Knight BW. "Local Structure Theory for Cellular Automata." *Physica D: Nonlinear Phenomena* 28, 1 (Sep 1987): 18-48.

Third, I am particularly interested in cellular automata not just as a case study in complexity, but, more importantly, as analogs to biology. The rules in Conway's game of life had clear biological counterparts: survival, death, birth, overpopulation, and isolation. I am confident that other neighborhoods and rulesets have biological analogs, but they are often hard to interpret. Hopefully, by categorizing rulesets and neighborhoods through math, patterns will emerge that yield information on the biological relevance of the system. I think of this sort of like a very qualitative principle component analysis.

*Note:* I might end up working on rtNEAT (real-time Neuroevolution of Augmenting Topologies) depending on how hard it is. I've seen some really interesting stuff with it and I'm sure it could be modeled using some sort of adaptive game theory. One more potential idea is a sort of ecological modeling (a combination of traditional Lotka-Volterra equations and game theory) in response to changes in climate and under the influence of humans as ecosystem engineers.

# 4 References

Chan, Bert WC. "Lenia: Biology of Artificial Life." *Complex Systems* 28, no 3 (2019): 251-286. https://doi.org/10.25088/ComplexSystems.28.3.251

Wolfram, Stephen. "Cellular Automata as Models of Complexity." *Nature* 311, no 4 (Oct 1984): 419-424. https://www.nature.com/articles/311419a0.pdf

Ermentrout GB and Edelstein-Keshet L. "Cellular Automata Approaches to Biological Modeling." *Journal of Theoretical Biology* 160 (1993): 97-133.

Gardner, Martin. "The Fantastic Combinations of John Conway's New Solitaire Game 'Life'." *Scientific American* 223 (Oct 1970): 120-123. https://web.stanford.edu/class/sts145/Library/life.pdf

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Rendell, Paul. "A Fully Universal Turing Machine in Conway's Game of Life." *Journal of Cellular Automata* 8, no. 1 (2013): 19-38.

Slackermanz. "Understanding Multiple Neighborhood Cellular Automata." May 23, 2021. https://slackermanz.com/understanding-multiple-neighborhood-cellular-automata/

Von Neumann, John. *Theory of Self-Reproducing Automata.* Published by *University of Illinois Press* (Urbana: 1966).

# 5 Links

Cellular Automata Wikipedia: https://en.wikipedia.org/wiki/Cellular_automaton

Lenia Github: https://chakazul.github.io/lenia.html

Slackermanz's Website: https://slackermanz.com

Sebastian Lague's Github on MNCA: https://github.com/SebLague/MN-Cellular-Automata

TataSZ Github Page on CA Modifications: https://tatasz.github.io/compound_ca/