CONTINUOUS-STRATEGY EVOLUTIONARY GAMES IN STRUCTURED POPULATIONS USING WIRING DIAGRAMS

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Project Proposal

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1. Introduction

Evolutionary game theory is a vast subfield of evolutionary dynamics which models the interactions of species through dynamical systems using game theory. Within evolutionary game theory, there are further specializations of games with weak or strong selection, mutation, drift, populations with graph structure, populations with group structure, and finite- and continuous-strategies.

Perhaps the most general frameworks of evolutionary games are those on populations with structure, introduced by [LHN05], as games without structure arise as well-mixed cases of structured populations. As such, we narrow our focus to the subfield of evolutionary graph theory.

Within evolutionary graph theory, one of the most commonly considered games is the Prisoner's Dilemma or donation game. In this game, there are two actions, cooperating C or defecting D, with the payoff matrix A from Equation 1.1, where $b > c > 0$ are real parameters.

$$A = \begin{array}{cc} & \begin{array}{cc} \mathrm{C} & \mathrm{D} \end{array} \\ \begin{array}{c} \mathrm{C} \\ \mathrm{D} \end{array} & \begin{bmatrix} b-c & -c \\ b & 0 \end{bmatrix} \end{array} \tag{1.1}$$

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A comprehensive summary of the dynamics of this game in scenarios of weak and strong selection dynamics with continuous-strategies on two players, contiuous-strategies with well-mixed (finite- or infinite-) populations, and in finite-strategy structured populations, particularly the subset of these problems where the interaction and reproduction graphs are equal, can be found in [Now06]. This subset of results naturally raise the challenge of describing the dynamics of weak and strong selection with continuous-strategies in structured populations.

To answer this question, [ZKT12] analyzed continuous-strategy evolutionary dynamics under strong selection on graphs playing the donation game, finding largely contrasting empirical results from discete-strategies on graphs. In particular, continuous-strategies allowed for statistically significantly more frequent evolution of cooperating strategies than finite-strategies, suggesting differing equilibria of the systems.

In the case of weak selection, [ZZL11] found continuous-strategy analogues for many discrete-strategy results in the donation game framework on regular graphs, in particular the result that if $b/c > k$, where k is the degree of the regular population graph and assumed to be much less than the population size, then cooperation is favored asymptotically. These theoretical results were found to hold empirically on non-regular graphs, where k instead signifies the average degree of the population graph.

Despite the positive empirical results of [ZKT12] and [ZZL11] for the donation game on general population graphs and continuous-strategies, there is no noted approach to generalize the theoretical approaches from regular graphs to the more general graphs considered in finite-strategy games. We hope to provide one such framework for theoretical analysis of general evolutionary games on arbitrarily structured populations with continuous-strategies.

In particular, the work of [SVS20] has introduced a sheaf theoretic methodology of formalizing so-called *open dynamical systems*, a generalization of the dynamical systems considering in most evolutionary dynamics literature where the dynamical system may be exposed to "inputs" that affect the dynamics and exposes "outputs" to other dynamical systems. In this language of "inputs" and "outputs", it is easy to then describe "systems of dynamical systems" where inputs are provided by outputs of other systems. As an illustrative example [JM21], the Lotka-Volterra equations can be formed as an open dynamical system.

An additional benefit of the sheaf theoretic approach to open dynamical systems in [SVS20] is the notion of "synchronized systems", where a discrete system can be related to a continuous system by correlating their time clocks. This makes the wiring diagrams of synchronized systems a promising framework to analyze general continuous-strategy evolutionary games in structured populations, a goal we describe in further detail in Section 2.

2. Model

We describe three different leading approaches to open dynamical systems, each of which could prove to be a fruitful avenue of research for evolutionary graph dynamics in generalized finite and infinite strategies. The three leading theories are those of (see referenced works for notation and definitions)

- (Ti) Polynomial functors and the category Poly to describe categories and dynamics [SN21],
- (Tii) Time categories C (for discrete systems, $\tilde{\mathsf{Int}}_{\mathbb{N}}$ and for continuous systems, $\tilde{\mathsf{Int}}_{\mathbb{R}}$), a synchronized time topos $\tilde{\mathsf{Int}}/\,\mathrm{Sync}$, wiring diagram categories W_{C}, wiring diagram algebras $\mathsf{W}_{\mathsf{W}} \to \mathsf{Cat}$, machines $\mathsf{Spn}_{\mathsf{C}} : \mathsf{W}_{\mathsf{C}} \to \mathsf{Cat}$, and synchronized machines $\mathsf{Mch}_{\mathrm{Sync}} : \mathsf{W}_{\tilde{\mathsf{Int}}/\,\mathrm{Sync}} \to \mathsf{Cat}$ to describe sheaf theoretic open dynamical systems [SVS20],
- (Tiii) Bundle categories Bun, bundle doctrines (Bun, T), lenses of bundle categories $\mathsf{Lens}_{\mathsf{Bun}}$, interfaces of bundles Bun in the interface category $\mathsf{Interface}$, and the category of bundle doctrine dynamical systems $\mathsf{Dyn} : \mathsf{Interface} \to \mathsf{Cat}$ to develop covariant and contravariant theories of open dynamical systems [JM21].

We note that all of these theories are connected, as the topoi $\tilde{\mathsf{Int}}$ and $\tilde{\mathsf{Int}}/\,\mathsf{Sync}$ as well as the machines $\mathsf{Spn}_{\mathsf{C}}$ and $\mathsf{Mch}_{\mathsf{Sync}}$ constructed in (Tii) [SVS20] are embedded in the polynomial functors and category Poly of (Ti) [SN21], and in turn the covariant and contravariant bundle doctrines (Bun, T) and resulting dynamics Dyn of (Tiii) are too contained within Poly of (Ti).

Each of these theories is quite verbose and intricate, and so we present now only a heuristic guiding approach to modeling evolutionary graph dynamics as open dynamical systems, and leave the specifics of each theory's model to future work as noted in Section 3.

In essence, an open dynamical system (in the notation of [JM21]) can be characterized as having input variables I, output variables O, internal state S, a readout map $r : S \to O$, and an update map $u : I \times S \to S$. A wiring diagram (in the sense of [SN21] and [SVS20]) consists of many open dynamical subsystems who's inputs through time are dictated by some subset of the other open subsystems' outputs. A graph structure can then clearly be developed into a wiring diagram, where the wiring is determined by the edges, if the subsystems' readout and update dynamics are specified (as well as their inputs, outputs, and internal state). In the setting of evolutionary graph theory, we aim to model each individual in the population as an open subsystem, with internal state equal to their strategy, update dictated by reproduction, mutation, and selection dynamics (such as the quasispecies equation), and readout dictated by their fitness.

We note that an added benefit of the theory (Ti) available in wiring diagrams is the ability to model changes in the wiring through time, i.e. changes in the graph/group structure of the evolutionary graphs, and thus could help to provide a suitable analogue to drift.

3. Future Work

This hierarchy of theories suggest several approaches that could be taken to generalize evolutionary graph dynamics to open dynamical systems, as one may note that the inclusions of (Tii) and (Tiii) within (Ti) would make (Ti) the most natural choice for a more general formalization of open dynamical systems on evolutionary graphs, but in some sense loses the granularity of synchronized machines of (Tii) or the specification of covariant and contravariant systems provided by (Tiii).

To balance this tradeoff, we make note that (Tiii), while the least general, provides a result that the open dynamical systems may be described as a form of "matrix arithmetic" akin to the well-understood approaches currently taken in evolutionary graph dyanmics. As such, (Tiii) seems an appropriate theory in which to initially generalize evolutionary graph dynamics to open dynamical systems.

With this formalization in mind, then (Tii) provides an accesible template to synchronizing finite and infinite open dynamical systems, and to some extent characterizes the relationships between finite and infinite systems. We hope that the formalization from (Tiii) could be bootstrapped with (Tii) to provide continuous time analogues of the well-characterized finite evolutionary graph dynamics that are currently known.

Finally, (Tiii) as the most general would be the final step in a full formalization of evolutionary graph dynamics through open dynamical systems.

At each steps of these increasingly general theories and formalizations, we hope that we could reproduce the known results of evolutionary graph dynamics in the various formalizations, showing the soundness of the formalization. After showing this soundness, we hope the more robust theories could provide currently unknown results.

We also make final mention Poly of (Ti) and bundles of (Tiii) has historically been developed in an effort for improving the speed of numerical approximations of large systems with repeated subparts, much like those observed in evolutionary graph dynamics, and would hope to explore these efficient approximations as empirical justification of our theorization.

This proposed plan would hopefully illuminate evolutionary graph theory with the added tools of open dynamical systems, or in failing could also characterize short-comings of open dynamical systems as currently formalized.

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