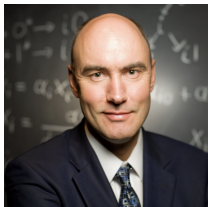


Evolutionary Graph Theory

How population structure affects evolutionary dynamics

David Brewster

Harvard University



Martin Nowak



Josef "Pepa" Tkadlec

Timeline



- ▶ 1809: Lamarck publishes *Philosophie zoologique*; Darwin born

Timeline



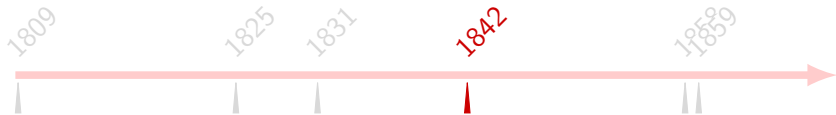
- ▶ 1825-31: Darwin student in Edinburgh and Cambridge

Timeline

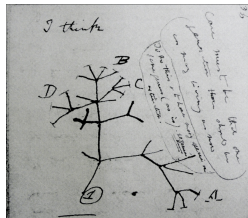


- ▶ 1831-36: *Voyage of the Beagle*

Timeline



- ▶ 1842: Pencil sketch



Timeline



- ▶ 1858: Darwin and Wallace publish on natural selection

Timeline



- ▶ 1859: *On the origin of species*

Evolutionary graph theory

How does stuff propagate through networks?

Maximizing the spread of influence through a social network

D Kempe, J Kleinberg, E Tardos - ... of the ninth ACM SIGKDD international ..., 2003 - dl.acm.org

Models for the processes by which ideas and influence propagate through a social network have been studied in a number of domains, including the diffusion of medical and technological innovations, the sudden and widespread adoption of various strategies in game-theoretic settings, and the effects of "word of mouth" in the promotion of new products. Recently, motivated by the design of viral marketing strategies, Domingos and Richardson posed a fundamental algorithmic problem for such social network processes: if we can try to ...

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[HTML] Complex networks: Structure and dynamics

S Boccaletti, V Latora, Y Moreno, M Chavez... - Physics reports, 2006 - Elsevier

Coupled biological and chemical systems, neural networks, social interacting species, the Internet and the World Wide Web, are only a few examples of systems composed by a large ...

☆ Save ⓘ Cite Cited by 11953 Related articles All 42 versions

Statistical physics of social dynamics

C Castellano, S Fortunato, Y Loreto - Reviews of modern physics, 2009 - APS

... best to mention relevant **social** science literature and highlight connections to it, the main focus of this work remains a description of the **statistical physics** approach to **social dynamics**. ...

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[HTML] Evolutionary dynamics on graphs

E Lieberman, C Hauert, MA Nowak - Nature, 2005 - nature.com

... Here we introduce **evolutionary graph** theory, which suggests a promising new lead in the effort to provide a general account of how population structure affects **evolutionary dynamics**. ...

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[HTML] Evolutionary games on graphs

G Szabó, G Fath - Physics reports, 2007 - Elsevier

Game theory is one of the key paradigms behind many scientific disciplines from biology to behavioral sciences to economics. In its **evolutionary** form and especially when the ...

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[HTML] Statistical physics of human cooperation

M Perc, JJ Jordan, DG Rand, Z Wang, S Boccaletti... - Physics Reports, 2017 - Elsevier

... the relevance of **physics** in all of this. Methods of **statistical physics** have recently been ... **Statistical physics of social dynamics** [13], of evolutionary games in structured populations [...]

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- ▶ coronavirus among humans
- ▶ influence (opinion, gossip, fake news) on social media
- ▶ genetic mutation in a population of individual organisms

Evolutionary graph theory

Evolutionary biology 101:

- ▶ Evolution acts on populations
- ▶ Individuals: bacteria in a Petri dish, cells in a tissue, birds in an archipelago, ...
- ▶ Individuals **reproduce** (sexually/asexually) and die
- ▶ type = genetic information, alters the **fitness**

Two main forces:

1. **mutation**: generates variety
2. **selection**: reduces variety

When mutations are rare: What is the fate of a single new mutant?

Model: Moran process on a graph

[HTML] Evolutionary dynamics on graphs

E.Lieberman, C.Hauert, M.A.Nowak - Nature, 2005 - nature.com

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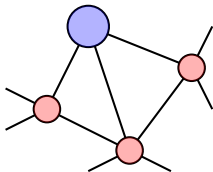
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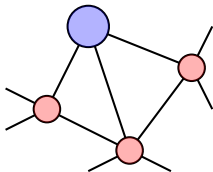
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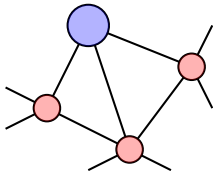
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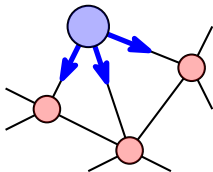
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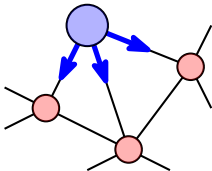
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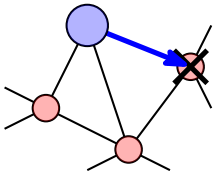
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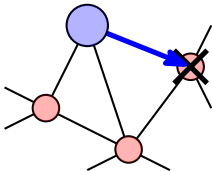
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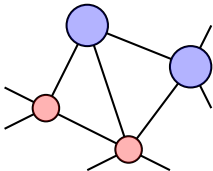
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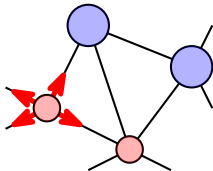
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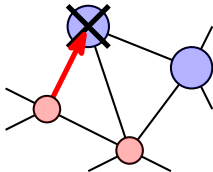
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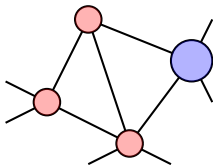
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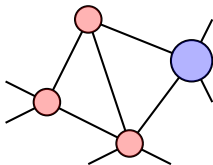
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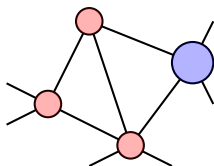
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Some features of the Moran process



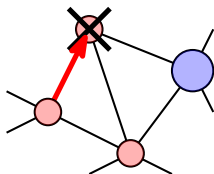
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1. It is stochastic (random).
2. In some steps, nothing happens.
3. Nodes can toggle back and forth (more opinions than gossip).
4. Eventually, all nodes become the same type (no mutation).
5. Variants exist (e.g. death-Birth updating).

Quantities of interest:

1. **Fixation probability $fp^r(G)$** : Average probability that, starting from a single node, mutants spread to all sites.
2. **Fixation time $T^r(G)$** : Average time until one type wins.
 - ▶ Measured in **steps** or better in **generations**.

Some features of the Moran process



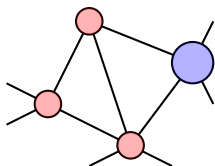
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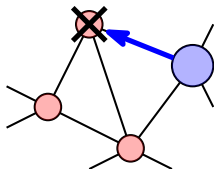
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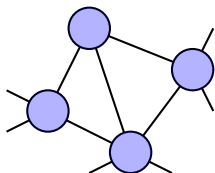
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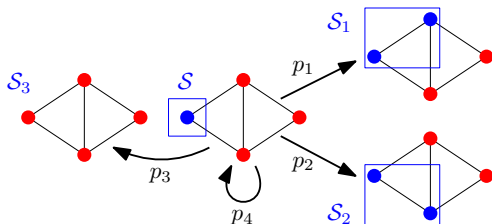
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Quantities of interest:

1. **Fixation probability $fp^r(G)$** : Average probability that, starting from a single node, mutants spread to all sites.
2. **Fixation time $T^r(G)$** : Average time until one type wins.
 - ▶ Measured in **steps** or better in **generations**.

Computing $\text{fp}^r(G)$

Claim. $\text{fp}^r(G)$ can be computed by solving a system of 2^n linear equations.

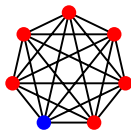


Proof. Suppose G has n nodes. There are 2^n configurations of nodes occupied by mutants. For each configuration S , let $\text{fp}^r(G, S)$ be the fixation probability if mutants initially occupy S . Then

$$\text{fp}^r(G, S) = \sum_{S'} p_{S \rightarrow S'} \cdot \text{fp}^r(G, S')$$

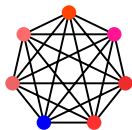
and $\text{fp}^r(G, \emptyset) = 0$, $\text{fp}^r(G, [n]) = 1$.

Special case: $r = 1$



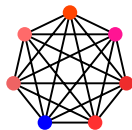
Claim. $\text{fp}^r(K_n) =$

Special case: $r = 1$



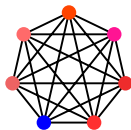
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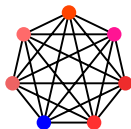
On any G , when $r = 1$ then fixation probability is **additive**, that is,

$$\text{fp}^r(G, \mathcal{S}) = \sum_{v \in \mathcal{S}} \text{fp}^r(G, \{v\}).$$

→ It suffices to solve a system of n equations.

→ Also, $\text{fp}^r(G) = 1/n$ for any graph G_n on n nodes.

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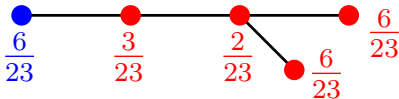
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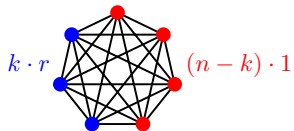
→ It suffices to solve a system of n equations.

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Claim. When G is **undirected** then $\text{fp}^r(G, \{v\}) \propto 1/\deg(v)$.



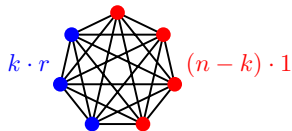
Special case: Complete graph K_n and $r > 1$



$$F = kr + (n - k)$$

$$\left. \begin{aligned} p_k^+ &= \frac{kr}{F} \cdot \frac{n-k}{n-1} \\ p_k^- &= \frac{n-k}{F} \cdot \frac{k}{n-1} \end{aligned} \right\} \frac{p_k^+}{p_k^-} = r$$

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It turns out that we are always r -times more likely to gain than to lose a mutant. Thus the process can be mapped to a 1-dimensional random walk, with a constant forward bias r .



Claim. $\text{fp}^r(K_n) = \frac{1-1/r}{1-1/r^n} \rightarrow_{n \rightarrow \infty} \boxed{1 - 1/r}$.

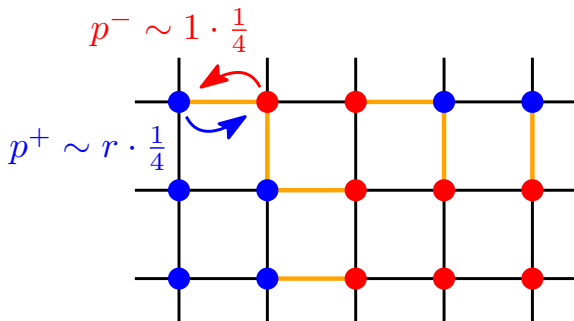
(**Intuition.** Let $x = 1 - \text{fp}^r(K_n)$. Then $x = \frac{1}{r+1} \cdot 1 + \frac{r}{r+1} \cdot x^2$.)

Special case: Regular graphs R_n

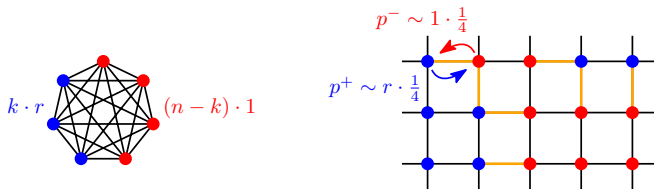
Claim (Isothermal theorem, '05). For any regular graph we have

$$\text{fp}^r(R_n) = \text{fp}^r(K_n).$$

Proof. The same mapping works! We say that an edge is **active** if its endpoints are of different types. Each active edge is r -times more likely to be used in gaining rather than losing a mutant.



But #steps on regular graphs differ



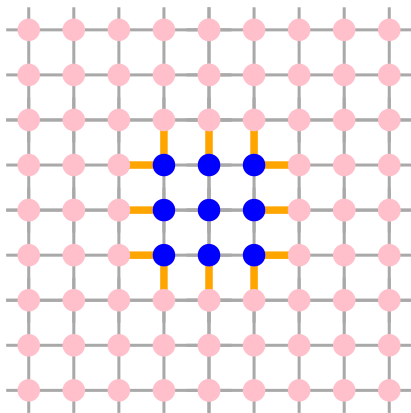
Intuition. If a of E edges are active, then, on average, roughly one in every E/a steps is active.



K_n : #edges $\sim n^2$, #active edges $\sim k(n-k)$
 $\rightarrow$ #steps for $K_n \sim c \cdot n + \sum_k \frac{n^2}{k(n-k)} \sim \Theta(n \log n)$

Sq_n : #edges $\sim 4n$, #active edges $\in (\sqrt{k}, 4k)$
 $\rightarrow$ #steps for Sq_n is $\mathcal{O}(n\sqrt{n})$ and $\Omega(n \log n)$.

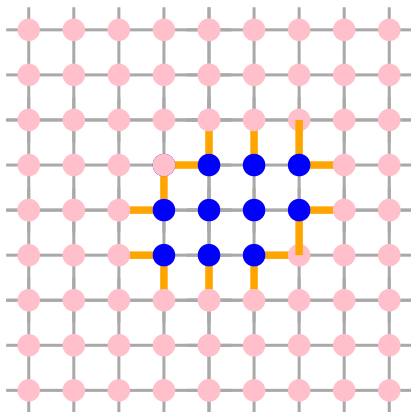
#steps on large square grids



Intuition. Most of the time, the boundary should have size $\sqrt{k}$, so

$$\text{\#steps for Sq}_n \sim c \cdot n + \sum_k \frac{n}{\sqrt{k}} \sim n \cdot \sqrt{n}.$$

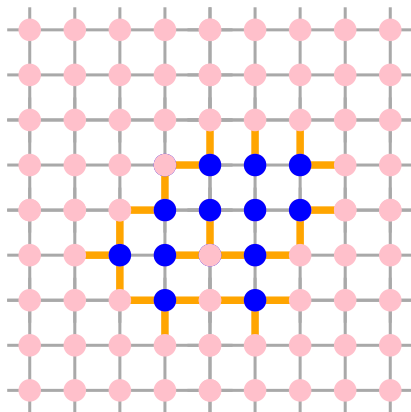
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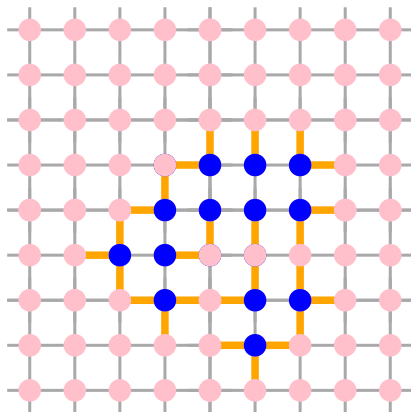
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Theorem. [DGRS '16] #steps for d -regular graph G is at most

$$\frac{2 \cdot d \cdot n \cdot H_n}{i(G) \cdot \text{fp}_{\min}^r(G)}$$

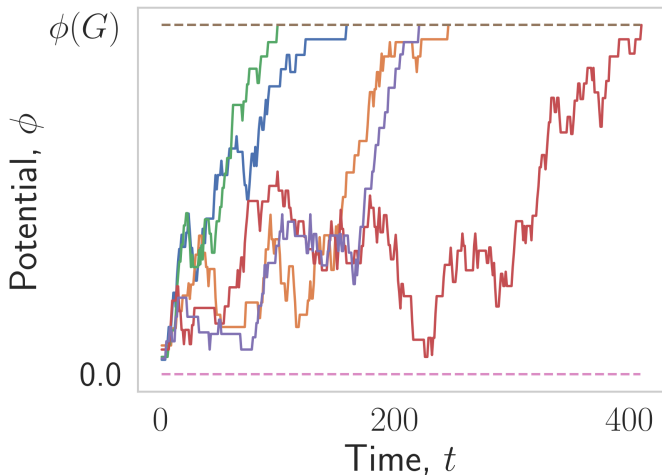
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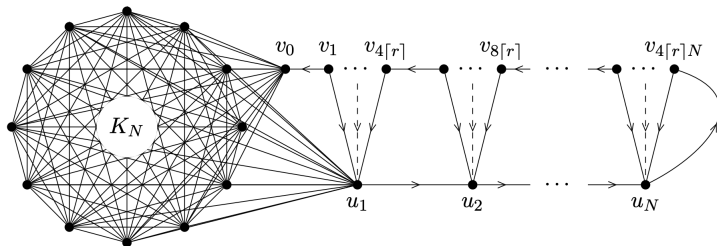
$$\frac{\phi(G) - \mathbb{E}[\phi(X_0)]}{\text{fp}_{\min}^r(G)} \cdot \frac{n^3}{1-1/r}$$

Directed graphs

Some directed graphs have exponentially long fixation time

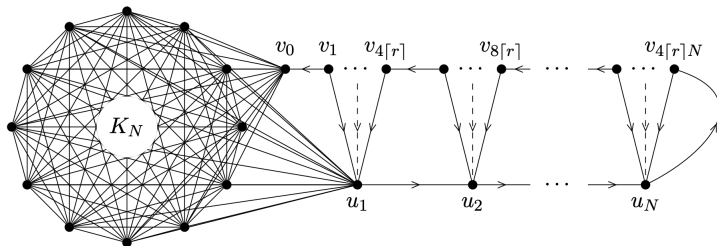
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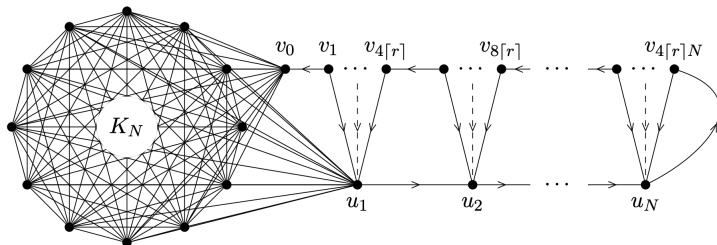
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Theorem. [BNT '23+] For *Eulerian* digraph with min degree δ and max degree Δ ,

$$r \geq \Delta/\delta \implies \# \text{steps at most } \text{poly}(n).$$

Approximating fixation probabilities in directed graphs

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Intuition. When $r = 1$, fixation probabilities satisfy

$$\text{fp}^{r=1}(G, \{v\}) \sum_{u: u \rightarrow v \in E} \frac{1}{\deg^+(u)} = \frac{1}{\deg^+(v)} \sum_{w: v \rightarrow w \in E} \text{fp}^{r=1}(G, \{w\})$$

for each $v \in V$.

Then a coupling and averaging argument gives a bound on #steps with $r \geq 1$ using only quantities involving $r = 1$.

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Consider similar model expect:

- ▶ residents never reproduce
- ▶ “density” constraints

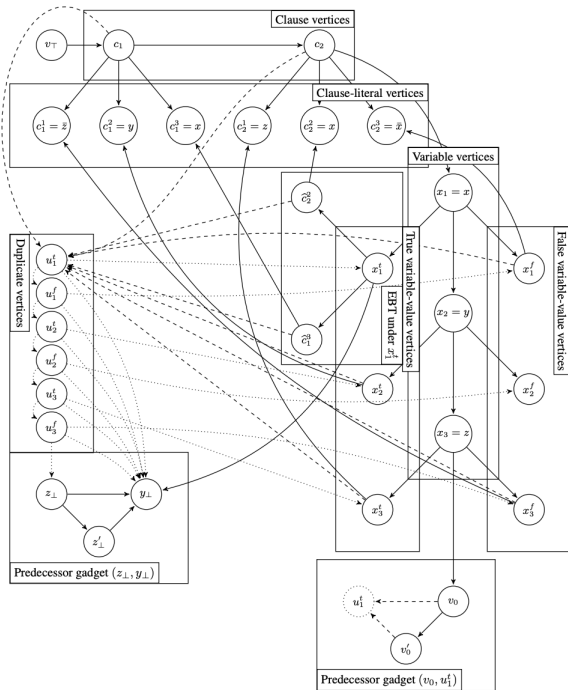
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Theorem. [ICN '15]

- ▶ Deciding whether $\text{fp}^r(G) > 0$ is NP-COMplete
- ▶ Approximating $\text{fp}^r(G)$ is #P-COMplete



Open questions

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- ▶ Tighter bounds on complexity of computing fixation probabilities
- ▶ What does “time” mean if population size is not constant?

Questions?