This PSET should not be turned in.

1 (100 PTS.) THE HAWK-DOVE GAME

Consider the following scenario from *Evolution and the Theory of Games* (Smith 1982):

> Imagine that two animals are contesting a resource of value V. By "value", I mean that the Darwinian fitness of an individual obtaining the resource would be increased by V. Note that the individual which does not obtain the resource need not have zero fitness. Imagine, for example, that the "resource" is a territory in a favourable habitat, and that there is adequate space in a less favourable habitat in which losers can breed. Suppose, also, that animals with a territory in a favourable habitat produce, on average, 5 offspring, and that those breeding in the less favourable habitat produce 3 offspring. Then V would equal $5 - 3 = 2$ offspring. Thus V is the gain in fitness to the winner, and losers do not have zero fitness. During the contest an animal can behave in one of three ways: `display`, `escalate`, and `retreat`. An animal which `displays` does not injure its opponent; one which `escalates` may succeed in doing so. An animal which `retreats` abandons the resource to its opponent. In real contests, animals may switch from one behaviour to another in a complex manner. For the moment, however, I suppose that individuals in a given contest adopt one of two "strategies"; for the time being, I assume that a particular individual always behaves in the same way.
>
> - HAWK: `escalate` and continue until injured or until opponent `retreats`.
> - DOVE: `display`; `retreat` at once if opponent `escalates`.
>
> If two opponents both `escalate`, it is assumed that, sooner or later, one is injured and forced to retreat. Alternatively, one could suppose that both suffer some injury, but for the moment I am seeking the simplest possible model. Injury reduces fitness by a cost, C.

1.A. Write down the 2×2 HAWK-DOVE game payoff matrix using the following assumptions.
- HAWK vs HAWK: Each contestant has a 50% chance of injuring its opponent and obtaining the resource, V, and a 50% chance of being injured.
- HAWK vs DOVE: HAWK obtains the resource, and DOVE retreats before being injured.
- DOVE vs DOVE: The resource is shared equally by the two contestants.

1.B. In plain English, describe what a Nash equilibrium is.

1.C. Identify any pure-strategy Nash equilibria in the above game. How do they depend on V and C?

1.D. (**Hard.**) Consider the following ***mixed*** strategy, denoted HAWK-DOVE(p): the individual plays as a HAWK with probability p and as a DOVE with the remaining probability. Consider the average possible payoffs between each of these mixed strategies. Identify any mixed-strategy Nash equilibria in this game.

1.E. Suppose there is a population of all DOVEs. Then in a rare event, an ε fraction of the population now consists of HAWKs (for example, due to a mutation). Recall that the payoff matrix from **1.A.** denotes gain in reproductive fitness. If a DOVE plays against a uniformly selected random individual of the population, what is the DOVE's expected fitness gain payoff?

1.F. Now suppose there is a population of all HAWKs. Then in a rare event, an ε fraction of the population now consists of DOVEs. Recall that the payoff matrix from **1.A.** denotes gain in reproductive fitness. If a HAWK plays against a uniformly selected random individual of the population, what is the HAWK's expected fitness gain payoff?

1.G. A strategy $\mathcal{S}$ is called ***evolutionarily stable*** (ESS) if when almost all members of the population adopt $\mathcal{S}$, then the fitness of these typical members is greater than that of any possible mutant. (Otherwise, the mutant could invade the population, and $\mathcal{S}$ would not be stable) (Smith 1982). In the HAWK-DOVE game which consists of only two strategies (i.e. these are the only possible mutants), are there any evolutionarily stable strategies? What conditions are necessary? (**Hint:** Use the solutions to the previous two questions and take $\varepsilon \longrightarrow 0$.)

1.H. What is a relation between a Nash equilibrium and an ESS? Is one more broad than the other? Or are they not comparable?

Solution:

1.A. If a HAWK plays another HAWK, they each have an expected payoff of $V/2 - C/2$. If a HAWK plays a DOVE, the hawk gets a payoff of V and the DOVE gets a payoff of 0. If a DOVE plays a DOVE, they each have a payoff of $V/2$. So the payoff matrix (indexed HAWK, DOVE in that order along both the rows and columns) is

$$\begin{pmatrix} (V-C)/2 & V \\ 0 & V/2 \end{pmatrix} \tag{1}$$

where the payoff is to the strategy on the row.

1.B. It is a scenario where no player can gain a higher payoff by unilaterally changing their strategy, assuming the strategies of the other players remain unchanged. Another way to view it is in terms of stability. Under a certain notion of stability, a Nash equilibrium is a stable point.

1.C. If $V > C > 0$ then both players playing as $\textsc{Hawk}$s is a Nash equilibrium.

1.D. Here is a sketch of one way to figure out the solution. We need to determine which values of p and q make it such that if one player plays as $\textsc{Hawk-Dove}(p)$ and the other as $\textsc{Hawk-Dove}(q)$, they are each incentivized to keep their same strategy. This means that

(a) for any $p' \neq p$, the expected payoff (to the strategy with parameter p or p') between $\textsc{Hawk-Dove}(p)$ and $\textsc{Hawk-Dove}(q)$ is higher than the expected payoff between $\textsc{Hawk-Dove}(p')$ and $\textsc{Hawk-Dove}(q)$

(b) for any $q' \neq q$, the expected payoff (to the strategy with parameter q or q') between $\textsc{Hawk-Dove}(p)$ and $\textsc{Hawk-Dove}(q)$ is higher than the expected payoff between $\textsc{Hawk-Dove}(p)$ and $\textsc{Hawk-Dove}(q')$.

When computing the expected payoff of these games, there should be four summands with coefficients pq, $p(1-q)$, $(1-p)q$, and $(1-p)(1-q)$. The coefficients are multiplied by entries in the payoff matrix.

1.E. A $\textsc{Dove}$ will play against a $\textsc{Hawk}$ with probability ε and will play against a $\textsc{Dove}$ with the remaining probability. So a $\textsc{Dove}$'s expected payoff in this population is

$$\varepsilon \cdot 0 + (1-\varepsilon) \cdot V/2. \tag{2}$$

1.F. A $\textsc{Hawk}$ will play against a $\textsc{Dove}$ with probability ε and will play against a $\textsc{Hawk}$ with the remaining probability. So a $\textsc{Hawk}$'s expected payoff in this population is

$$\varepsilon \cdot V + (1-\varepsilon) \cdot (V-C)/2. \tag{3}$$

1.G. (a) Suppose the population is all $\textsc{Dove}$s and then in a rare event, an ε fraction of the population consists of $\textsc{Hawk}$s due to a mutation. From the solution to **1.E.**, a $\textsc{Dove}$ has expected fitness change

$$(1-\varepsilon) \cdot V/2. \tag{4}$$

A $\textsc{Hawk}$ has expected fitness change

$$\varepsilon(V-C)/2 + (1-\varepsilon) \cdot V. \tag{5}$$

So $\textsc{Dove}$ is an ESS when (4) is larger than (5) as $\varepsilon \to 0$. But this is only true as $\varepsilon \to 0$ when $V/2 > V$. That would mean we would need $V < 0$ for $\textsc{Dove}$ to be an ESS. If $V = 0$, then $\textsc{Dove}$ is an ESS when $(V-C)/2 < 0$, which happens when $C > V$.

(b) Suppose the population is all $\textsc{Hawk}$s and then in a rare event, an ε fraction of the population consists of $\textsc{Dove}$s due to a mutation. From the solution to **1.F.**, a $\textsc{Hawk}$ has expected fitness change

$$\varepsilon(V-C)/2 + (1-\varepsilon) \cdot V. \tag{6}$$

A $\textsc{Dove}$ has expected fitness change

$$\varepsilon V/2 + (1-\varepsilon) \cdot 0. \tag{7}$$

So $\textsc{Hawk}$ is an ESS when (6) is larger than (7) as $\varepsilon \to 0$. This is true as $\varepsilon \to 0$ when $V > 0$.
Suppose $V = 0$ Then $\textsc{Hawk}$ is an ESS when $(V-C)/2 > V/2$, which happens when $C < V$.

Thus in the typical scenario of $V > C > 0$, only $\textsc{Hawk}$ is an ESS.

1.H. Every ESS is a Nash Equilibrium. Here are some more notes on the relationships between concepts of stability in games.

2 (100 PTS.) ITERATED GAMES

Recall the Prisoner's Dilemma setup from last lecture.

2.A. What are the conditions of the payoff matrix that make the game a prisoner's dilemma?

Now consider an *iterated* game like we saw in the last lecture.

2.B. Consider the possible pure (i.e. deterministic) strategies that take in all of the information from the last k iterations of the game. These are known as ***memory-k*** pure strategies. How many such strategies are there?

2.C. Now, consider the possible pure strategies that take in only the information about the opponent's actions in the past k rounds of the game. These are known as ***reactive*** memory-k pure strategies. How many such strategies are there?

2.D. How do the answers of **2.B.** and **2.C.** compare?

2.E. For each of the answers for **2.B.** and **2.C.**, which values of k do the number of strategies start to exceed 10^{80} (approximately the number of hadrons in the observable universe)?

For the iterated Prisoner's Dilemma game, consider the following mixed strategy parameterized by p and q: if opponent cooperated last round, cooperate with probability p; if opponent defected last round, cooperate with probability q.

2.F. Is this a memory-k strategy? If so, what is k?

2.G. Is it reactive?

2.H. For the strategies we saw last lecture, what are the appropriate parameters (if it is even possible)? Remember we saw ALWAYS COOPERATE, ALWAYS DEFECT, TIT-FOR-TAT, GENEROUS TIT-FOR-TAT, WIN-STAY LOSE-SHIFT, and possibly more.

Solution:

2.A. Suppose you and I are playing the game and the payoff matrix is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{8}$$

where a is the payoff you get if we both cooperate, b is the payoff you get if you cooperate and I defect, c is the payoff you get if you defect and I cooperate, and d is the payoff you get if we both defect. Then the game is a prisoner's dilemma if and only if $c > a > d > b$.

2.B. Suppose you and I are playing the game. Each deterministic memory-k strategy takes as input $2k$ moves (since we are looking at your moves and my moves) and outputs one move. Each move is either `cooperate` or `defect`. Thus we are counting the number of functions of the form

$$\mathcal{S}\colon \{\texttt{cooperate}, \texttt{defect}\}^{2k} \to \{\texttt{cooperate}, \texttt{defect}\}. \tag{9}$$

In general, if we have two finite sets A and B with sizes $|A|$ and $|B|$, the number of possible functions from A to B is $|B|^{|A|}$. Thus there are $2^{2^{2k}}$ such memory-k strategies.

2.C. Now we are looking for the number of functions of the form

$$\mathcal{S}_{\text{reactive}}\colon \{\texttt{cooperate}, \texttt{defect}\}^{k} \to \{\texttt{cooperate}, \texttt{defect}\}. \tag{10}$$

Using similar reasoning as the previous question, we get that there are 2^{2^k} such reactive memory-k strategies.

2.D. Let us find the value of x such that $(2^{2^k})^x = 2^{2^{2k}}$. Taking $\log_2$ of both sides gives

$$2^k x = 2^{2k}. \tag{11}$$

Thus $x = 2^{2k}/2^k = 2^{2k-k} = 2^k$. In other words, the number of deterministic reactive memory-k strategies is only a 2^kth root of the number of deterministic memory-k strategies. This is a huge reduction of the strategy space!

In general, if we have set A' such that $|A'| = \sqrt{|A|}$, Then the number of functions from A' to B is $|B|^{|A'|} = |B|^{\sqrt{|A|}}$.

2.E. For all deterministic memory-k strategies, we want to find the smallest integer k such that $2^{2^{2k}} > 10^{80}$. Taking $\log_2$ of both sides gives $2^{2k} > 80\log_2 10$. Taking another $\log_2$ of both sides and dividing by two gives

$$k > \frac{1}{2}\log_2(80) + \frac{1}{2}\log_2\log_2 10 = \log_2\sqrt{80} + \frac{1}{2}\log_2\log_2 10 \approx 0.86 + \log_2 9 \approx 4. \tag{12}$$

For reactive memory-k strategies, we want to find the smallest integer k such that $2^{2^k} > 10^{80}$. Taking $\log_2$ of both sides gives $2^k > 80\log_2 10$. Taking another $\log_2$ of both sides gives

$$k > \log_2(80) + \log_2\log_2 10 \approx 6.3 + 1.7 \approx 8. \tag{13}$$

In general, by restricting to reactive strategies, we can evaluate strategies with memory twice as long as the unrestricted space before the number of possible strategies becomes "too large" to mathematically analyze.

2.F. We need k to be at least 1 because the strategy is observing the past round. But we do not need k to be larger than 1. So we can say that this is a memory-1 strategy. Technically you could also say this is a memory-k strategy for any $k \geqslant 1$, but we will focus on the smallest k.

2.G. Yes, it is reactive because we are only looking at the opponent's actions.

2.H.
- Always Cooperate: $p = 1$, $q = 1$, we start with cooperation
- Always Defect: $p = 0$, $q = 0$, we start with defection
- Tit-for-tat: $p = 1$, $q = 0$, and we start with cooperation first
- Generous Tit-for-tat: $p = 1$, q is a parameter for these strategies. we start with cooperation first
- Win-stay Lose-shift: This is not a reactive strategy.

3 (100 PTS.) THE EVOLUTION OF TRUST

Play the following game: https://ncase.me/trust/.