Worksheet 2 Solutions

For each problem below, read the instructions carefully. Use Expert Methodology. Prioritize process and learning over finding the "correct" numerical answer.

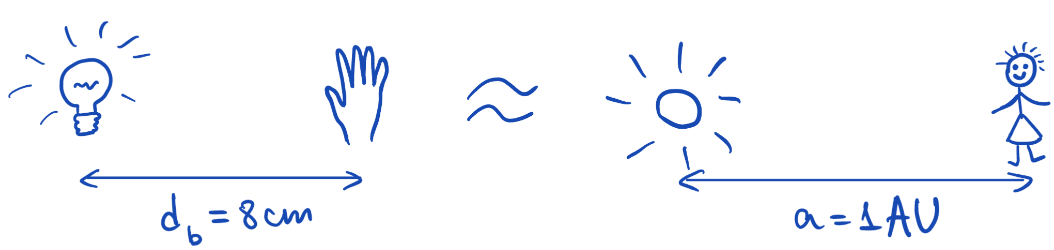
1. Consider the amount of energy produced by the Sun per unit time, also known as the **luminosity**, $L_\odot$. That same amount of energy per time is present at the surface of all spheres centered on the Sun at distances $d > R_\odot$. However, unlike the luminosity, the **flux**, the energy per unit time per unit area, depends on the distance away from the Sun.

   (a) How does flux, F, depend on luminosity, L, and distance, d?

   Flux has units of ergs/sec/cm^2, and Luminosity has units of ergs/sec. Therefore, from the dimensional analysis we can conclude that $L \sim \frac{F}{d^2}$.
   The luminosity is the same at the surfaces of all spheres centered on the source. But the area of each sphere is $4\pi d^2$, so the energy per time par area is spread out more with increasing distance d. Therefore,

   $$F = \frac{L}{4\pi d^2}$$

   (b) When the back of your hand is about 8cm away from a 100-Watt incandescent light bulb, the bulb feels like the Sun on a sunny, Spring day. The Earth-Sun distance is 1 astronomical unit (AU), or $a = 1.5 \times 10^{13}$ cm. Use this information to estimate the luminosity (power output) of the Sun ($L_\odot$) in units of ergs per second. (1 Watt is 10^7 ergs/sec)



   We feel flux, therefore $F_b = F_\odot$. Using the relation between luminosy and flux:

   $$\frac{L_b}{4\pi d_b^2} = \frac{L_\odot}{4\pi a^2}$$

   .
   Solving it for the Solar Luminosity:

$$L_0 = \frac{4\pi a^2}{4\pi d_b^2} \cdot L_b = \frac{a^2}{d_b^2} L_b = \left(\frac{1.5 \cdot 10^{13} \text{ cm}}{8 \text{ cm}}\right)^2 \left(100 \text{ W} \cdot \frac{\text{erg/sec}}{\text{W}}\right) = 3.5 \cdot 10^{33} \text{erg/sec}$$

(c) The Solar flux at 1 AU (1.5×10^{13} cm) is known as the Solar Constant. Calculate its value.

$$F_\odot = \frac{L_\odot}{4\pi a^2} = \frac{3.8 \cdot 10^{33} \frac{\text{erg}}{\text{s}}}{4\pi \left(1.5 \cdot 10^{13} \text{ cm}\right)^2} = 1.3 \cdot 10^6 \frac{\text{erg}}{\text{s cm}^2}$$

(d) Later this semester we'll learn about blackbody radiation. For now, take it as a given that the Sun radiates approximately as a blackbody, and as such its luminosity is related to its temperature and radius by

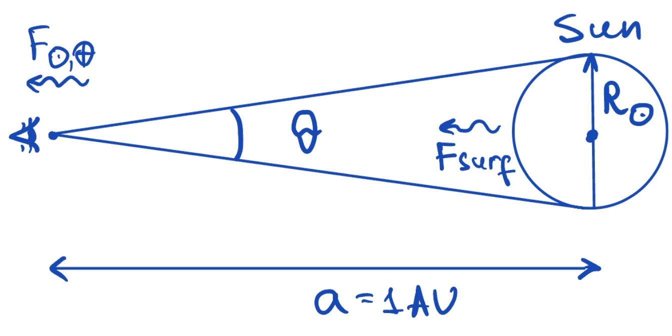
$$L_\odot = 4\pi R_\odot^2 F_{\text{surf}} = 4\pi R_\odot^2 \sigma T_\odot^4$$

where $F_{\text{surf}} = \sigma T_\odot^4$ is the flux of the Sun at its surface, and $\sigma = 5.7 \times 10^{-5}$ erg s^{-1} cm^{-2} K^{-4} is the Stefan-Boltzman constant. The Sun's angular diameter is $\theta = 0.5$ degrees. Use this information along with your estimate of the Solar Constant to estimate the temperature of the Sun.

(HINT: If you feel stuck, start with the seemingly trivial expression involving the luminosity of the Sun at two different distances:

$$L_\odot(d = R_\odot) = L_\odot(d = 1 \text{ AU}),$$

Discuss this equation in your group until it makes intuitive sense. Then expand the left and right sides in terms of their respective distances, d, and fluxes at those distances, until you can solve for $T_\odot$.)



We can write down Solar luminosity on different distances from the Sun:

$$L_\odot(d = R_\odot) = 4\pi R_\odot^2 F_{\text{surf}}$$

$$L_\odot(d = a = 1 \text{ AU}) = 4\pi a^2 F_{\odot,\oplus}$$

where $F_{\odot,\oplus}$ is flux from the Sun that we measure on the Earth, or Solar Constant.

Since luminosity is the property of the Sun, we can make these equations equal. Therefore,

$L_\odot = 4\pi R_\odot^2 F_{\text{surf}} = 4\pi a^2 F_{\odot,\oplus}$, or $4\pi R_\odot^2 \sigma T_\odot^4 = 4\pi a^2 F_{\odot,\oplus}$

Solving this equation for temperature:

$$T_\odot^4 = \frac{4\pi a^2 F_{\odot,\oplus}}{4\pi R_\odot^2 \sigma} = \frac{a^2 F_{\odot,\oplus}}{R_\odot^2 \sigma}$$

We can find radius of the Sun, using it's angular diameter θ.

$$R_\odot = a \sin\frac{\theta}{2} = a \cdot \frac{\theta}{2}$$

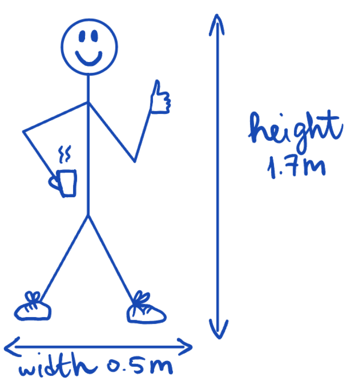
As a result, we have

$$T_\odot = \left(\frac{a^2 F_{\odot,\oplus}}{a^2\theta^2/4 \cdot \sigma}\right)^{1/4} = \left(\frac{4}{\sigma}\frac{F_{\odot,\oplus}}{\theta^2}\right)^{1/4}$$

Plugging in numbers, we can estimate temperature of the surface of the Sun $T_\odot = 5900K$

2. Assume that the Sun emits all of its light in the form of photons with a wavelength of 0.5 μm (microns = 10^{-4} cm).

   (a) Using your answer from the previous question, how many Solar photons hit your body each second if you are sunbathing on a Summer day? (Planck's constant is $h = 6.6 \times 10^{-27}$ erg· sec)



We receive energy that is equal to number of photons multiplied by their energy $\Delta E = \Delta E_\gamma \cdot N_\gamma$. Energy of photon can be written as $\Delta E_\gamma = h\nu = \frac{hc}{\lambda}$.

Therefore, we can write Solar flux as:

$$F_\odot = \frac{\Delta E}{\Delta t \Delta A} = \frac{N_{\gamma,\odot} \cdot hc}{\Delta t \Delta A \cdot \lambda}$$

Solving it for number of photons N_γ and using an approximation that human body is a rectangle, so $\Delta A = \text{width} \cdot \text{height}$:

$$N_{\gamma,\odot} = \frac{\Delta t \Delta A \cdot \lambda \cdot F_\odot}{hc} = \frac{\Delta t \cdot \text{height} \cdot \text{ width } \lambda F_\odot}{hc} =$$

$$= \frac{1\text{s} \cdot 170 \text{ cm} \cdot 50 \text{ cm} \cdot 0.5 \cdot 10^{-4} \text{ cm} \cdot 1.3 \cdot 10^6 \frac{\text{erg}}{\text{s cm}}}{(6.6 \cdot 10^{-27}\text{erg} \cdot \text{s})\,(3 \cdot 10^{10} \text{ cm/s})} = 3 \cdot 10^{21} \text{ photons per second}$$

(b) Alpha Centauri is a Sun-like star that is 4 light years away from the Earth. How many photons per second would hit your body if you are "starbathing" under Alpha Centauri one evening?

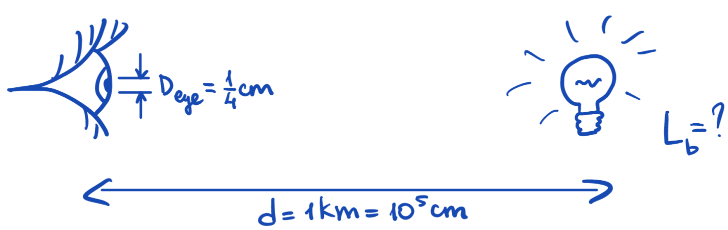
$$\frac{F_{\alpha\text{Cen}}}{F_{\odot}} = \left(\frac{L}{4\pi D_{\alpha\text{Cen}}^2}\right)\left(\frac{L}{4\pi a^2}\right)^{-1} = \left(\frac{a}{D_{\alpha\text{Cen}}}\right)^2$$
$$\frac{N_{\gamma,\odot}}{N_{\gamma,\alpha\text{Cen}}} = \frac{F_{\alpha\text{Cen}}}{F_{\odot}}$$

Therefore,

$$N_{\gamma,\alpha\text{Cen}} = \gamma,\odot \cdot \left(\frac{D_{\alpha\text{Cen}}}{a}\right)^2 = 3\cdot 10^{21}\cdot\left(\frac{4\cdot(3\cdot 10^8\ \text{cm})}{15\cdot 10^{13}\ \text{cm}}\right)^2 = 4.5\cdot 10^9 \text{ photons per second}$$

3. Let's do an order-of-magnitude problem. The goal is to find the answer accurate to a factor of 10 (1, 10, 100, or...?). Make your best estimates of various quantities needed to solve the problem.

The eye must receive $\sim$ 10 photons in order to send a signal to the brain that says, "Yep, I see that." If you are standing in an enormous, completely dark cave and just barely discern a light bulb at a distance of 1 kilometer, what is the power output of the bulb?
Note: Assume the bulb is emitting light isotropically.
Note: The energy of a photon is $E = h\nu$, for a frequency ν (Greek letter 'nu'), and Planck's constant $h = 6.6\times 10^{-27}$ erg s. The speed of light is 3×10^{10} cm s^{-1}. The visible part of the electromagnetic spectrum is at approximately 0.5 microns.
Note: Energy is not the same as power! Power and energy are related, though.



Flux of the bulb from the distance d is:

$$F_{\text{eye}} = \frac{L_b}{4\pi d^2}$$

We can estimate total amount of energy that eye received over exposure time Δt and over whole area of the pupil A_{eye}:

$$E_{\text{eye}} = F_{\text{eye}}\cdot A_{\text{eye}}\cdot\Delta t = \frac{L_b}{4\pi d^2}\cdot A_{\text{eye}}\cdot\Delta t$$

On the other hand, eye received N_γ of photons, therefore $E_{\text{eye}} = N_\gamma E_\gamma$

We can equal right hand sides of these formulas

$$\frac{L_b}{4\pi d^2}\cdot A_{\text{eye}}\cdot\Delta t = N_\gamma E_\gamma$$

And then solve it for luminosity of the bulb L_b:

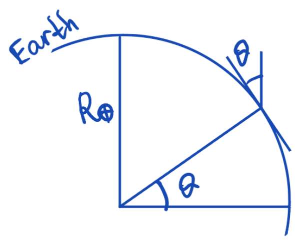
$$L_b = \frac{4\pi d^2 N_\gamma E_\gamma}{A_{\text{eye}} \cdot \Delta t} = \frac{16 d^2 N_\gamma hc}{D_{\text{eye}}^2 \cdot \Delta t \cdot \lambda} = \frac{16 N_\gamma hc}{\Delta t \cdot \lambda} \cdot \left(\frac{d}{D_{\text{eye}}}\right)^2$$

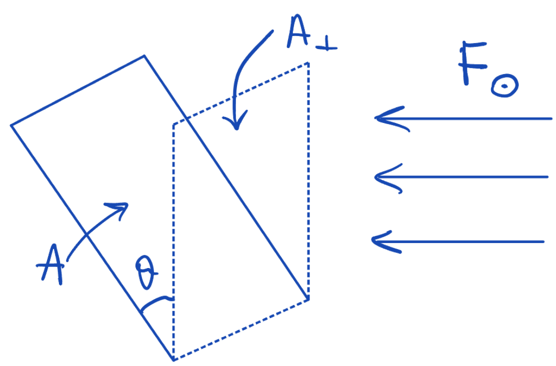
We know all the values except exposure time Δt. The eye is "read out" by the brain at some "frame rate" R_{eye}. Computer monitors refresh at about 60 Hz which is at least 2x the eye's frame rate did to the Nyquist Limit. Let's call $\frac{1}{\Delta t} = f_{\text{eye}} = 30$ Hz $= 30$ s^{-1}.
Therefore, $\Delta t = \frac{1}{f_{\text{eye}}} = 3 \cdot 10^{-2} s$.
Plugging in all the numerical values:

$$L_b = 700 \text{ erg/s} = 7 \cdot 10^{-5} \text{ W}$$

4. Let's think about the variation of temperature on the surface of the Earth.

   (a) Ignoring the tilt of the Earth and clouds, how does the flux received from the Sun on the surface of the Earth vary with latitude? (HINT: Flux cares about the direction of light with respect to the surface at which it is measured.)





   From the figure we can say that $A_\perp = A\cos\theta$. Therefore, since definition of the flux is $F = \frac{\Delta E}{A \Delta t}$, flux at the latitude θ is:

$$F_\theta = \frac{\Delta E}{A \Delta t} = \frac{\Delta E}{\frac{A_\perp}{\cos\theta}\Delta t} = \frac{\Delta E}{A_\perp \Delta t} \cdot \cos\theta = F_\odot \cos\theta$$

   (b) Now, factoring in the tilt of the Earth—which is constant with respect to its orbital plane, at least on time scales less than 1000 years—*qualitatively* explain why we experience variations in temperature at a fixed latitude during the period the Earth's orbit (a.k.a. seasons during the year).

   In the Northern winter the Earths surface is tilted away from the Sun by $\Delta\theta$ which increases the area by $(\cos(\theta + \Delta\theta))^{-1}$ and decreases the received flux, making the average temperature less.

5. Many students consider the astronomical magnitude system for measuring stellar fluxes baffling and a bit scary. With this exercise, I'd like to convince you that while it is somewhat baffling, but not necessarily scary. Magnitudes are defined such that, for Star One and Star Two with fluxes F_1 and F_2,

$$\frac{F_1}{F_2} = 10^{0.4(m_2 - m_1)},$$

where m_1 is the magnitude of Star One and m_2 is the magnitude of Star Two. Notice that, to two significant figures, $10^{0.4} \approx 2.51$. Note also that a larger magnitude corresponds to a *fainter* star. Using this information, fill out the following table, where $\Delta m = m_2 - m_1$, and identify the simple (approximate to **2 sig figs**, but use $10^{0.4} = 2.51$ to three sig figs before rounding) recursion relation:

Δm	$F_1/F_2 = 2.51^{\Delta m}$	F_1/F_2 (rounded)
0	1	1
1	2.51	2.5
2	6.30	6.3
3	15.8	16
4	39.7	40
5	99.63	$1 \cdot 10^2$
6	250.1	$2.5 \cdot 10^2$
7	627.6	$6.3 \cdot 10^2$
8	1575	$16 \cdot 10^2$
9	3954	$40 \cdot 10^2$
10	9925	$1 \cdot 10^4$
...	...	...

Worksheet 2 — Homework 1

Homework Questions: The following problems are to be completed primarily on your own, making use of what you've learned in class and from working on the previous questions with your group. You may receive hints and assistance from your classmates and the teaching staff. But the expectation is that the solutions you turn in is a reflection of your learning.

1. Suppose you are observing two stars, $\alpha\,$CMa (α Canis Majoris, aka Sirius) and $\lambda\,$Sco (λ Scorpii, aka Shaula). $\lambda\,$Sco is 3 magnitudes fainter than $\alpha\,$CMa. (You may use subscripts α and λ for $\alpha\,$CMa and $\lambda\,$Sco, respectively.)

   (a) Which star has the higher magnitude? Which star is brighter?
   Since $\lambda\,$Sco is 3 magnitudes fainter than $\alpha\,$CMa, therefore $m_{\lambda\,\mathrm{Sco}} = m_{\alpha\,\mathrm{CMa}} + 3$ and Sirius ($\alpha\,$CMa) is brighter

   (b) How does $\lambda\,$Sco's flux compare to that of $\alpha\,$CMa's?

$$\frac{F_{\lambda\,\mathrm{Sco}}}{F_{\alpha\,\mathrm{CMa}}} = 10^{0.4(m_{\alpha\,\mathrm{CMa}} - m_{\lambda\,\mathrm{Sco}})} = 10^{0.4\cdot(-3)} = 10^{-1.2} \approx \frac{1}{16}$$

   (c) How much longer do you need to observe $\lambda\,$Sco to collect the same number of photons with your detector as you do for $\alpha\,$CMa?

$$F = \frac{\Delta E}{\Delta t \Delta A} = \frac{N_\gamma \cdot h\nu}{\Delta t \Delta A} \sim \frac{N_\gamma}{\Delta t}$$

   Therefore, $\Delta t \sim \frac{N_\gamma}{F}$
   If $N_{\gamma,\lambda\,\mathrm{Sco}} = N_{\gamma,\alpha\,\mathrm{CMa}}$, then $\frac{\Delta t_{\gamma,\lambda\,\mathrm{Sco}}}{\Delta t_{\gamma,\alpha\,\mathrm{CMa}}} = \left(\frac{F_{\gamma,\lambda\,\mathrm{Sco}}}{F_{\gamma,\alpha\,\mathrm{CMa}}}\right)^{-1}$.

$$\Delta t_{\gamma,\lambda\,\mathrm{Sco}} = 16 \Delta t_{\gamma,\alpha\,\mathrm{CMa}}$$

   We need to observe $\lambda\,$Sco 16 times longer.

2. Imagine a star that is 100 light years away that you can barely see from a dark site at night.

   (a) What is the star's power output? Assume the star emits all of its light in the visible range ($\lambda = \mu$m). Express your answer both in units of ergs per second, and in units of Solar luminosity $L_\odot$. Feel free to draw on your answers to previous questions on this worksheet!
   Flux of the star from the distance d is:

$$F_{\mathrm{eye}} = \frac{L_\star}{4\pi d_\star^2},$$

   where $d = 100$ lyr $= 100 \cdot c \cdot 1$ yr $= 9.46 \cdot 10^{19}$ cm.
   We can estimate total amount of energy that eye received over exposure time Δt and over whole area of the pupil A_{eye}:

$$E_{\mathrm{eye}} = F_{\mathrm{eye}} \cdot A_{\mathrm{eye}} \cdot \Delta t = \frac{L_\star}{4\pi d_\star^2} \cdot A_{\mathrm{eye}} \cdot \Delta t$$

   On the other hand, eye received N_γ of photons, therefore $E_{\mathrm{eye}} = N_\gamma E_\gamma$
   We can equal right hand sides of these formulas and solve it for $L_\star$:

$$L_\star = \frac{4\pi d_\star^2 N_\gamma E_\gamma}{A_{\mathrm{eye}} \cdot \Delta t} = \frac{4\pi d_\star^2 N_\gamma hc}{\frac{\pi D_{\mathrm{eye}}^2}{4} \cdot \Delta t \cdot \lambda}$$

We plug in numbers:

$$L_\star = \frac{16(9.46 \cdot 10^{19}\ \text{cm})^2 \cdot 10 \cdot \left(6.6 \cdot 10^{-27}\text{erg} \cdot \text{s}\right)\left(3 \cdot 10^{10}\ \text{cm/s}\right)}{(0.5\ \text{cm})^2 \cdot 3 \cdot 10^{-2}\ \text{s} \cdot 5 \cdot 10^{-5}\ \text{cm}} = 7 \cdot 10^{32}\ \text{erg/s}$$

To find luminosity in Solar luminosity, we divide by $L_\odot$:

$$L_\star = \frac{7 \cdot 10^{32}}{4 \cdot 10^{33}} L_\odot = 0.19 L_\odot$$

(b) What is the star's apparent magnitude? The Sun has an apparent magnitude of $m_\odot = -26.8$. Look up the human eye's limiting magnitude (assuming perfectly dark sky conditions), and compare your answer to what you calculate. What reasons can you think of for any significant difference in these values?

To estimate star's apparent magnitude, we can write an equation for the star and the Sun:

$$m_2 - m_1 = -2.5 \log \frac{F_2}{F_1} \Rightarrow m_\star - m_\odot = -2.5 \log \frac{F_\star}{F_\odot}$$

Therefore, we need to find $\frac{F_\star}{F_\odot}$:

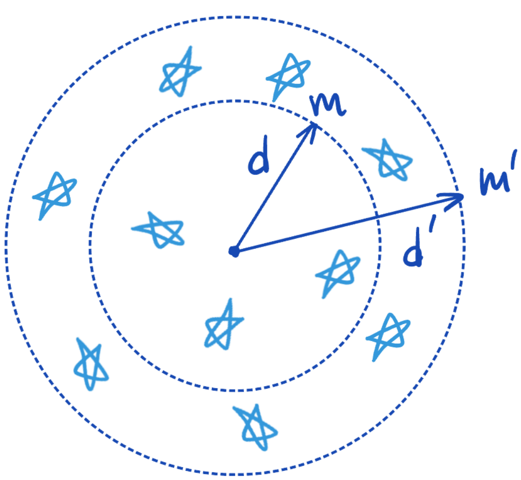
$$\frac{F_\star}{F_\odot} = \left(\frac{L\star}{4\pi d_\star^2}\right)\left(\frac{L_\odot}{4\pi a^2}\right)^{-1} = \frac{L_\star}{L_\odot} \cdot \frac{a^2}{d_\star^2}$$

Now we combine it with the first equation and plug in numbers from the previous part of the problem:

$$m_\star = m_\odot - 2.5 \log\left(\frac{L_\star}{L_\odot} \cdot \frac{a^2}{d_\star^2}\right) = -26.8 - 2.5 \log\left(0.19 \cdot \frac{(1.496 \cdot 10^{13}\ \text{cm})^2}{(9.46 \cdot 10^{19}\ \text{cm})^2}\right) = 9$$

The limiting magnitude of the eye is about 8 mag, which is a factor of 2.5 lighter than this estimate. The difference can be caused by a higher number of photons needed to be detected by the eye (25 instead of 10), or sky conditions.

3. Assume that the part of the Galaxy that we live in is uniformly filled with Sun-like stars with a number density $n_\star$. Your survey is designed to observe every Sun-like star down to an apparent magnitude m. How many more stars can we observe if we redesign the survey to observe down to a magnitude $m + 1$? In other words, compare the number of stars you can observe with the original design to the design that has a limiting magnitude that is one magnitude fainter



If we observe down to a new magnitude limit of $m' = m - 1$, we observe out to a new distance d'. Assume all stars have a luminosity L.

We can find the flux ratio of stars at distances d and d':

$$\frac{F}{F'} = \frac{L}{4\pi d^2} \cdot \left(\frac{L}{4\pi d'^2}\right)^{-1} = \left(\frac{d'}{d}\right)^2$$

On the other hand, we can define this ratio, using magnitudes:

$$\frac{F}{F'} = 2.51^{(m-m')}$$

Then, we can combine these equations above to find the distance ratio:

$$\frac{d'}{d} = \left(\frac{F}{F'}\right)^{1/2} = 2.51^{(\mathrm{m-m'})/2}$$

Now, we can estimate, how many more stars are located within that bigger radius d'. The larger distance corresponds to larger volume.

$$\frac{V'}{V} = \frac{\frac{4}{3}\pi d^{13}}{\frac{4}{3}\pi d^3} = \left(\frac{d'}{d}\right)^3$$

Since stars are distributed uniformly, $n_\star = \frac{N}{V} = \frac{N'}{V'}$. Therefore,

$$\frac{N'}{N} = \frac{V'}{V} = \left(\frac{d'}{d^3}\right)^3 = \left(\frac{F}{F'}\right)^{3/2} = 2.51^{\frac{3}{2}(m-m')} = 2.51^{\frac{3}{2}(m-(m-1))} = 2.51^{3/2} = 4$$

4. **Coding Practice:** Create a plot of $\sin(2\pi\omega t)$ vs time, t, for three choices of the frequency, ω, with three different line colors (use colors that provide good contrast!). Label your axes and label each curve with "ω = [value]" positioned above each curve.

```
import numpy as np
import matplotlib.pyplot as plt
t = np.linspace(0, 1, 101)
w = [1, 2, 3]
plt.rcParams['text.usetex'] = True
fig = plt.figure(figsize=(8, 5))

plt.plot(t, np.sin(2 * np.pi * w[0] * t), lw=3, c='tab:red',
    label=fr'$\omega = $ {w[0]}')
plt.plot(t, np.sin(2 * np.pi * w[1] * t), lw=3, c='tab:green',
    label=fr'$\omega = $ {w[1]}')
plt.plot(t, np.sin(2 * np.pi * w[2] * t), lw=3, c='tab:blue',
    label=fr'$\omega = $ {w[2]}')

plt.xlabel('$t$', fontsize=20)
plt.ylabel(r'$sin(2 \pi \omega t)$', fontsize=20)
plt.legend(fontsize=15)
plt.tick_params(axis='both', which='major', labelsize=12)
plt.grid('--')
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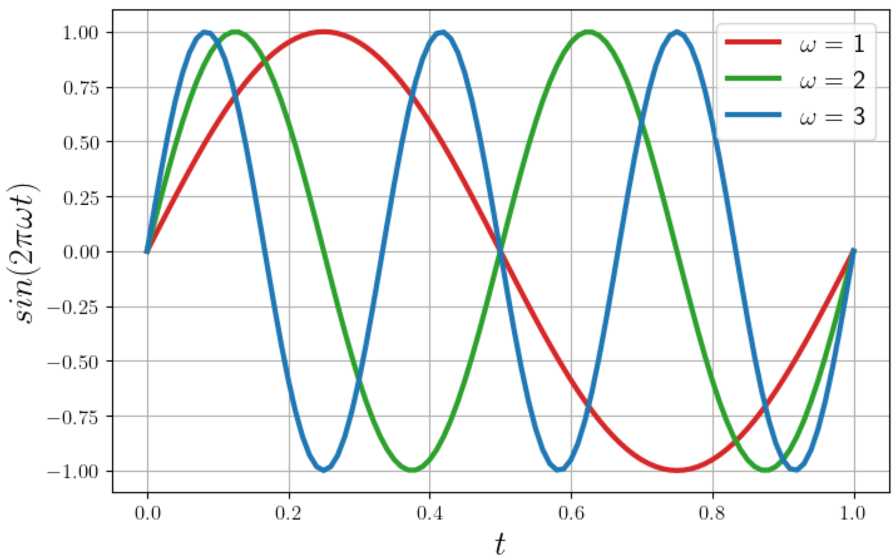

$\omega = 1$
$\omega = 2$
$\omega = 3$
$sin(2\pi\omega t)$
t
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0.0
0.2
0.4
0.6
0.8
1.0