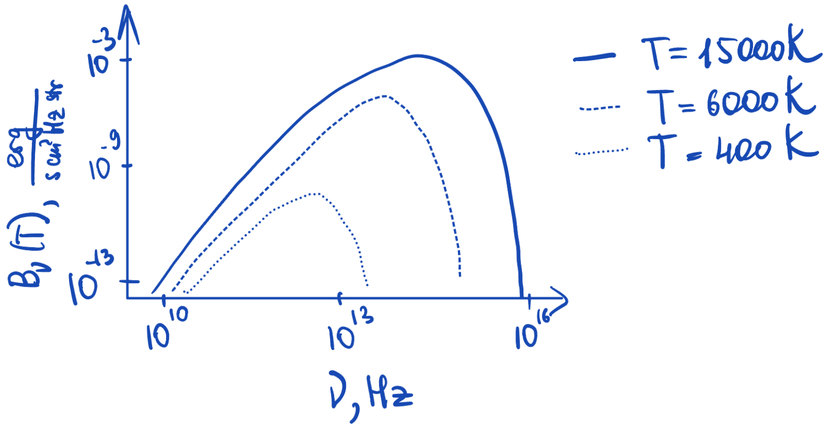
Worksheet 3 Solutions

1. A blackbody is a perfect emitter of thermal radiation, such that the properties of the emitted light is dependent only on the temperature of the object. The intensity of radiation emitted from a blackbody is given by

$$I_\nu(T) = \frac{2\nu^2}{c^2}\frac{h\nu}{\exp\left(\frac{h\nu}{kT}\right)-1} \equiv B_\nu(T) \tag{1}$$

$I_\nu(T)$ is the energy per time, per area, per frequency (ν), per solid angle (we'll deal with that solid angle down below. The units are radians^2). The Boltzmann's constant $k = 1.4 \times 10^{-16}$ ergs K^{-1} (ergs per Kelvin), the speed of light $c = 3 \times 10^{10}$ cm s^{-1}, and $h = 6.6 \times 10^{-27}$ erg $\cdot\, s$.

(a) Use your favorite plotting software to create plots of $B_\nu(T)$ versus ν (for $\nu > 0$) at two different values of the temperature, T. Play around with different values of T to gain some familiarity with the functional form of $B_\nu(T)$. **Recreate these plots at the board. Do the plots with different values of T ever cross?**



(b) The second term on the right hand side is measured in units of energy ($h\nu$). However, check out the units of the first term: cm^{-2}. The units of $B_\nu(T)$ are therefore: ergs per area. This is a strange set of units for the *flow* of energy! What about time? What about the frequency of light? Does this term have the correct units? Compare the units in the equation to the units in the sentence above describing $B_\nu(T)$.

The $h\nu$ term has units of energy (ergs). $\frac{\nu^2}{c^2}$ has units of cm^{-2}, or $\frac{1}{\text{Area}}$. This naively looks like $\frac{\text{Energy}}{\text{Area}}$ but the "per frequency, per time" part of the intensity definition is unitless together. Thus, $B_\nu(t)$ has units of energy per area, per time, per frequency so it all works out.

(c) $B_\nu(T)$ is measured per solid angle. Solid *what?* you may ask. A solid angle is a differential element of the surface of a sphere, measured in spherical coordinates θ and ϕ, and expressed as $d\Omega$. The units are steradians (pronounced "stair-radians"), but they are technically unitless in cgs, like radians.

i. Estimate the number of steradians subtended by your thumb held at arm's length.
The "rule of thumb" is that the thumb is 0.5 degrees wide at arm's length, or $\theta = 0.5 \text{ deg} \times \frac{\pi}{180} \approx 9 \times 10^{-3}$ radians. My thumb is about 3θ tall, so $\Omega = W \times H = \theta \times 3\theta = 3\theta^2 \approx \boxed{2 \times 10^{-4} \text{ sr}}$

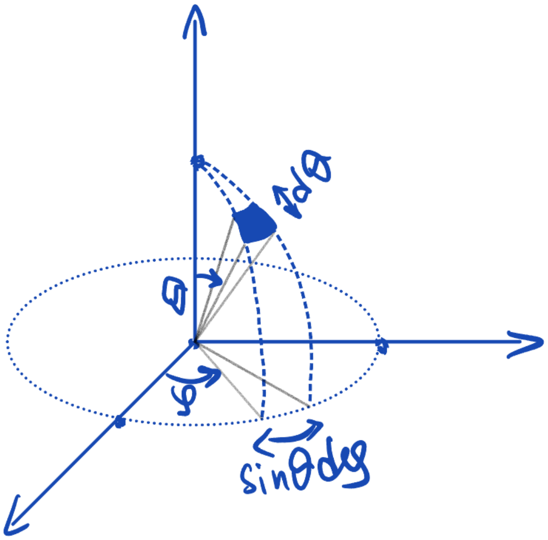
ii. Estimate the number of steradians subtended by the moon.
The moon has a surface area of $\approx 3.8 \times 10^7$ km^2 and is at a distance of $\approx 3.8 \times 10^5$ km (can be thought as the mean orbital radius about the Earth). The solid angle, $\Omega = \frac{A}{d^2}$, is the 2D angle subtended by a cross-sectional area A at a distance d from the point of observation. The surface area of the moon is equal to $4\pi r^2$ so the cross-sectional area of the moon is then $A = \pi r^2$ $(3.8/4) \times 10^7 = 9.5 \times 10^6$ km^2. Thus, $\Omega = \frac{9.5\times10^6}{(3.8\times10^5)^2} = \boxed{6.6 \times 10^{-5} \text{ sr}}$

iii. How many steradians is the entire night sky, from horizon to horizon (HINT: Think about the surface area of a sphere)
A solid angle is defined by the ratio of the surface area covered on a sphere by an object to the area given by the square of the radius of said sphere:

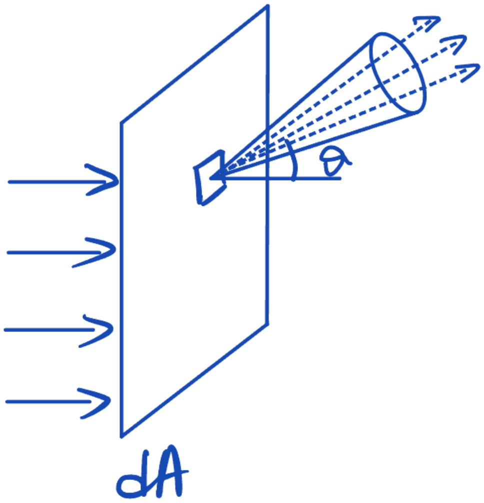
$$\Omega = \frac{A}{r^2}$$

The night sky, from horizon to horizon, is a hemisphere. The total surface area of a sphere is $A = 4\pi R^2$ so by definition the solid angle it subtends at the center of a sphere is $\frac{A}{R^2} = \frac{4\pi R^2}{R^2}$ which equals 4π steradians. The night sky is half of this, or $\boxed{\Omega_{Hem} = 2\pi \text{ sr}}$

(d) Draw a diagram illustrating this differential element of solid angle. Express $d\Omega$ as a trig function of θ and ϕ. (HINT: lines of fixed longitude are all the same length. Lines of fixed latitude have different lengths.)



(e) The photons flowing outward isotropically from a blackbody correspond to radiation with an *intensity* at a *specific* frequency, $B_\nu(T)$. This is the "specific intensity."
Draw a diagram illustrating this concept of energy flowing into a solid angle, per time through a surface, measured at a specific frequency. Convince your group that you understand what's going on with this diagram and concept. This is a key astrophysical concept and you should take this opportunity to talk with an instructor to make sure you understand it.



2. Now let's go from intensity to *flux* by cancelling out the "per steradian" part. How much radiation passes through a unit area dA on the surface of the blackbody, and into all solid angles? This quantity is F_ν, measured in units of energy per time, per frequency, per area (no steradians). Be careful that the unit-area, dA, needs to be projected onto a plane that is perpendicular to the direction of light travel, which leads to a term $dA\cos\theta$ instead of just dA. If your diagram in the previous question is correct, you should be able to reason out why this is so.

   The energy per time, per area, per frequency emerging from a Lambertian surface is at a maximum for rays normal to the surface, and decreases to zero for rays parallel to the surface, and varies as $\cos\theta$.

   $$F_\nu(T) = \int_0^{2\pi}\int_{-\pi}^{\pi} B_\nu(T)\cos\theta\sin\theta d\theta d\phi$$

   $B_\nu(T)$ has no angular dependence and can be factored out:

   $$F_\nu(T) = B_\nu(T)\int_0^{2\pi}\int_{-\pi}^{\pi} \cos\theta\sin\theta d\theta d\phi$$

   This simplifies to:

   $$\boxed{F_\nu(T) = \pi B_\nu(T)}$$

3. Let's now think about *specific luminosity*, L_ν, which is the monochromatic power output of an object (energy per time per frequency).

   (a) Assume the object is a star with a radius, $R_\star$, and integrate $F_\nu(T)$ over the entire surface of the star.

   $$\boxed{L_\nu = \left(\frac{8\pi^2 h\nu^3}{c^2}\right)\frac{R_*^2}{\exp(\frac{h\nu}{kT}) - 1}}$$

(b) Give an expression for L_λ.

Since $B_\lambda = \left|\frac{d\nu}{d\lambda}\right| B_\nu$ we know that:

$$L_\lambda = \left|\frac{d\nu}{d\lambda}\right| L_\nu = \frac{c}{\lambda^2} L_\nu$$

and $\nu = \frac{c}{\lambda}$, thus:

$$\boxed{L_\lambda = \left(\frac{8\pi^2 hc^2}{\lambda^5}\right) \frac{R_*^2}{\exp(\frac{hc}{\lambda kT}) - 1}}$$

(c) Suppose you are a distance, D, away from the star. How much flux do you receive from the star?

$$F_{\lambda,D} = \frac{L_\lambda}{4\pi D^2}$$

This becomes:

$$\boxed{F_{\lambda,D} = \left(\frac{2\pi hc^2}{\lambda^5 D^2}\right) \frac{R_*^2}{\exp(\frac{hc}{\lambda kT}) - 1}}$$

4. Blackbodies are nice because they're such simple objects. Their outward appearance is entirely determined by their temperature. If there were cows in space, astronomers would imagine them to be spherical blackbodies (and seriously, it wouldn't be a bad approximation). In this exercise we'll take advantage of the relative simplicity of blackbodies to derive some useful expressions that you'll use during this term, and throughout your astronomy career.

   (a) In astronomy, it is often useful to deal with something called the "bolometric flux," or the energy per area per time, independent of frequency. Integrate the specific flux $F_\nu(T)$ over all frequencies to obtain the bolometric flux emitted from a blackbody, $F(T)$. You can do this using by substituting the variable $u \equiv h\nu/kT$. This will allow you to split things into a temperature-dependent term, and a term comprising an integral over all frequencies. However, rather than solving for the integral, just set the integral and all constants equal to a new, single constant called σ, which is also known as the Stefan-Boltzmann constant. If you're really into calculus, go ahead and show that $\sigma \approx 5.7 \times 10^{-5}$ erg s^{-1} cm^{-2} K^{-4}. Otherwise, commit this number to memory.

$$F(T) = \int_0^\infty F_\nu(T)\, d\nu = \pi \int_0^\infty B_\nu(T) d\nu$$

Use u-substitution:

$$\nu = \frac{kT}{h} u$$

$$d\nu = \frac{kT}{h} du$$

$$F(T) = \int_0^\infty \frac{2\pi\nu^2}{c^2} \frac{h\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1} d\nu = \frac{2\pi k^4}{c^2 h^3} T^4 \int_0^\infty \frac{u^3}{e^u - 1} du = \frac{2\pi k^4}{c^2 h^3} \left(\frac{\pi^4}{15}\right) T^4$$

$$\boxed{F(T) = \sigma T^4}$$

(b) Write an expression for the total power output of a blackbody with a radius R, starting with the expression for $F(T)$ from the previous step. This total energy output per unit time is also known as the *bolometric luminosity*, L.

It's actually fine to start with the expression of $F = \sigma T^4$. We want power, or energy per time, and flux is energy per time per area, so we need to integrate out area at the surface of the blackbody with radius R:

$$L = \int_0^{4\pi R^2} F dA$$

$$L = 4\pi R^2 F = \boxed{4\pi R^2 \sigma T^4}$$

(c) Convert the units of the blackbody intensity from $B_\nu(T)$ to $B_\lambda(T)$ Note: Remember that the amount of energy in a frequency interval $d\nu$ has to be exactly equal to the amount of energy in the corresponding wavelength interval $d\lambda$.

There is equal energy in an interval $d\nu$ as there is in $d\lambda$.

$$B_\nu(T) d\nu = B_\lambda(T) d\lambda$$

Re-arranging for $B_\lambda(T)$:

$$B_\lambda(T) = \frac{d\nu}{d\lambda} B_\nu(T)$$

Note that the speed of light is related to its frequency and wavelength via $c = \lambda\nu$, thus $\nu = \frac{c}{\lambda}$. Taking the absolute value of the derivative of ν with respect to λ we get:

$$\left|\frac{d\nu}{d\lambda}\right| = \frac{c}{\lambda^2}$$

Thus,

$$B_\lambda(T) = \frac{c}{\lambda^2} B_\nu(T)$$

We know:

$$B_\nu(T) \equiv \frac{2\nu^2}{c^2} \frac{h\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1}$$

Plugging in $B_\nu(T)$ and $\nu = \frac{c}{\lambda}$ we get:

$$B_\lambda(T) = \left(\frac{c}{\lambda^2}\right)\left(\frac{2}{c^2}\right)\left(\frac{c^2}{\lambda^2}\right) \frac{hc/\lambda}{\exp(\frac{hc}{\lambda kT}) - 1}$$

This simplifies to:

$$\boxed{B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{\exp(\frac{hc}{\lambda kT}) - 1}}$$

(d) **The Wien Displacement Law:** Derive an expression for the wavelength $\lambda_{\max}$ corresponding to the peak of the intensity distribution at a given temperature T. (HINT: How do you find the maximum of a function? Once you do this, again substitute $u \equiv h\nu/kT$). The expression you end up with will be transcentental, but you can solve it easily to first order, which is good enough for this exercise.

The peak is given by $\frac{dB_\lambda}{d\lambda} = 0$, but we can simplify this by making a similar substitution to Exercise 4 (a). We can write:

$$u \equiv \frac{hc}{\lambda kT}$$

We can re-arrange this for λ:

$$\lambda = \frac{hc}{ukT}$$

Then plug in the above expression for λ into $B_u(T)$:

$$B_u(T) = \frac{2hc^2}{\left(\frac{hc}{ukT}\right)^5} \frac{1}{\exp(u) - 1}$$

This becomes:

$$B_u(T) = \frac{2hc^2}{h^5c^5} \frac{(ukT)^5}{\exp(u) - 1}$$

This simplifies to:

$$B_u(T) = \frac{2k^2T^5}{h^4c^3} \frac{u^5}{e^u - 1}$$

Ignoring the constants, we can solve for u_{max} in $\frac{dB_u(T)}{du} = 0$:

$$\frac{dB_u(T)}{du} = \frac{d}{du}\left[\frac{u^5}{e^u - 1}\right] = 0$$

We can type the above expression into Wolfram Alpha and tell it to solve for u. This gives us $u = 4.965 \approx 5$. Note, however, that a first order Taylor expansion gives us $u = 4$. Since $u = \frac{hc}{\lambda kT} = 5$ then:

$$\boxed{\lambda_{max} = \frac{hc}{5kT}}$$

(e) **The Rayleigh-Jeans Tail:** Next, let's consider photon energies that are much smaller than the thermal energy. Use a first-order Taylor expansion on the term $e^{\frac{hc}{\lambda kT}}$ to derive a simplified form of $B_\lambda(T)$ in this low-energy regime.

When photon energies are small compared to thermal energy:

$$\frac{hc}{\lambda} \ll kT$$

This allows us to Taylor expand $e^{\frac{hc}{\lambda kT}}$ around $\frac{hc}{\lambda kT} = 0$.
We can approximate:

$$e^{\frac{hc}{\lambda kT}} \approx \frac{hc}{\lambda kT} + 1$$

Then, $B_\lambda(T)$ becomes:

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{\exp(\frac{hc}{\lambda kT}) - 1} \approx \frac{2hc^2}{\lambda^5} \frac{1}{\frac{hc}{\lambda kT} + 1 - 1}$$

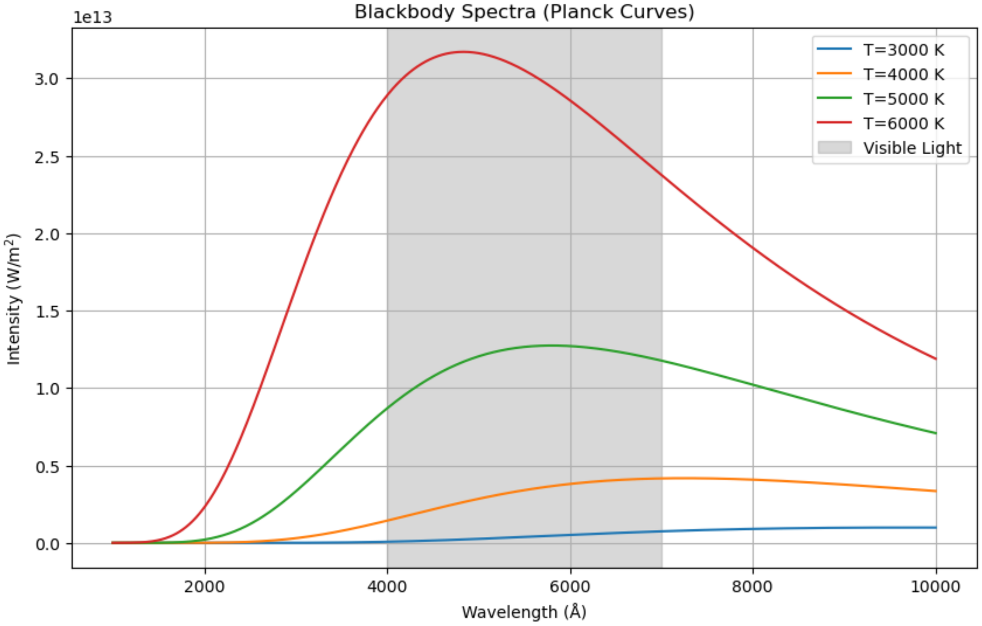
Thus,

$$\boxed{B_\lambda(T) \approx \frac{2ckT}{\lambda^4}}$$

The Planck function becomes a power law $B_\lambda(T) \sim \lambda^{-4}$

Homework 2

1. **Coding Practice:** Use Python to create a plot comparing the blackbody spectra (Planck curves) with temperatures of $T = \{3000, 4000, 5000, 6000\}$ K, expressed as intensity as a function of wavelength, $I_\lambda(T) = B_\lambda(T)$. Plot versus wavelength in Angstroms. Label each curve with its respective temperature. Indicate the band of wavelengths corresponding to visual light.



2. You observe two gravitationally bound stars (a binary pair). One is blue and one is yellow. The yellow star is six times brighter than the blue star. *Qualitatively* compare their temperature and radii, i.e. which is hotter, which is smaller? Next, *quantitatively* compare their radii (to 1 significant figure).

The bluer star is hotter. Hotter blackbodies are more luminous, unless a cooler star is much larger, since $L \sim R^2 T^4$. Since the yellow star is brighter, it must be much larger than the blue star. Now let's consider this quantiatively! The peak wavelength emitted by yellow light is approximately $\lambda_{max,Y} = 580$ nm. The peak wavelength emitted by blue light is approximately $\lambda_{max,B} = 470$ nm. The temperature which corresponds to the yellow light wavelength peak is:

$$T_Y = \frac{hc}{5k(\lambda_{max,Y})} \approx 5000K$$

The temperature which corresponds to the blue light wavelength peak is:

$$T_B = \frac{hc}{5k(\lambda_{max,B})} \approx 6000K$$

We know that the yellow star is six times brighter than the blue star:

$$L_Y = 6L_B = 6(4\pi R_B^2 \sigma T^4) = 4\pi R_Y^2 \sigma T_Y^4$$

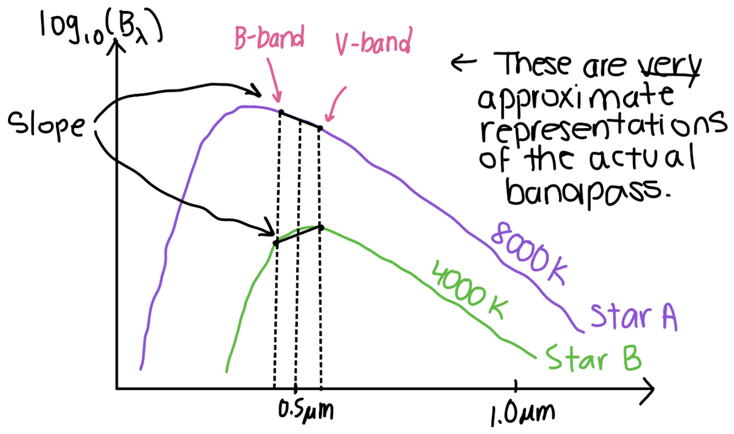
Re-arranging for the ratio between the stellar radii:

$$\frac{R_Y}{R_B} = \left[6\left(\frac{T_B}{T_Y}\right)^4\right]^{1/2}$$

This simplifies to:

$$\boxed{\frac{R_Y}{R_B} = 3.5 \approx 4}$$

3. So far we've been thinking about magnitudes in a bolometric sense. However, astronomers most frequently measure magnitudes in certain wavelength/frequency ranges, or "bandpasses," using standardized filters. Do a Google search for the Johnson B and V filters, and qualitatively explain why the quantity $B - V$, also known as color, is a proxy for a star's temperature. Go beyond yellow versus blue stars and Wien's law, and think about how the difference in B and V magnitudes changes as $B_\lambda(T)$ changes. Use illustrations and plots where appropriate.



Star A has a much higher temperature than Star B. The amount of energy radiated by Star A in the B and V filters is about equal such that $(B - V)_A = 0$. Star B emits less light in the B filter than in the V band, so it has $(B - V)_B > 1$. A star that is cooler than star B will have a larger B-V value, corresponding to a redder star. B-V color effectively measures the slope of the Planck curve at two wavelengths ($\approx 0.4\ \mu m$ and $\approx 0.5 \mu m$).

4. Prof. Johnson's former office mate, Mike Cushing (now a Prof. at U. of Toledo), discovered a new type of astrophysical object known as a "Y dwarf." Y dwarfs are a subclass of brown dwarfs that we'll learn about later in the term (and yes, astronomers say "dwarfs" as the plural of "dwarf"). Consider a Y dwarf residing near a Sun-like star (where near refers to a small angular separation between the two objects on the sky). Y dwarfs have a temperature of about 350 K (for reference, 0 degrees Celcius corresponds to 273 K), and as a result they are extremely faint. They're also very small, with a radius roughly the size of Jupiter's ($R_Y = R_{\rm Jup} \approx 0.1 R_\odot$).

   (a) At what wavelength, $\lambda_{\rm max}$, should you observe to have the best chance of detecting the Y dwarf?

   For a temperature of $T = 350$ K, the peak wavelength of light is at:

$$\lambda_{max} = \frac{hc}{5kT} \approx \boxed{\text{8 microns}}$$

   (b) As measured at $\lambda_{\rm max}$, how many photons per second, per cm^2 emitted from a Y dwarf at a distance of 30 light years would reach the *Spitzer* space telescope? Assume you observe over a narrow range of wavelengths such that $\Delta\lambda = 1\mu$m. HINT: Start with the expression for the specific luminosity, and be careful with the fact that L_λ is the power *per wavelength*.

There are two ways to go about this. The first starts with noting that we want to find $f$lux, energy per area, per second, where energy will be converted to the number of photons using $E = hc/\lambda$. We've been given a bandpass, so we'll specifically go for specific flux:

$$F_\lambda = \pi B_\lambda(T)$$

We can evaluate $B_\lambda(T)$ because we have the temperature of the Y dwarf, T_y. Next, note that we can go from specific flux to specific luminosity via

$$L_\lambda(R_\star) = 4\pi R_\star^2 F_\lambda(R_\star)$$

This expression of L_λ is evalated at the surface of the star, but we can evaluate it at a distance d just as easily:

$$L_\lambda(d) = 4\pi d^2 F_\lambda(d)$$

Now we can use $L_\lambda(d) = L_\lambda(R_\star)$ to get

$$F_\lambda(d) = F_\lambda(R_\star)\left(\frac{R_\star}{d}\right)^2 = \pi B_\lambda(T)\left(\frac{R_\star}{d}\right)^2$$

Finally, we evaluate B_λ at $\lambda_{\max}$ and $T_y = 350$ K, and integrate over the bandpass by simply multiplying by $\Delta\lambda$ to get

$$F(d) = \pi B_{\lambda_{\max}}(T_y)\left(\frac{R_\star}{d}\right)^2 \Delta\lambda$$

The second way of solving this problem is to notice, as before, that we want the flux at a distance, d, so

$$F(d) = \frac{dE}{dt\,dA} = \int_0^{\Omega_y}\int_{\lambda_{\max}}^{\lambda_{\max}+\Delta\lambda} B_\lambda(T)\,d\lambda\,d\Omega = B_\lambda(T)\cdot\Delta\lambda\cdot\Omega_y$$

where Ω_y is the solid angle into which the star is emitting in order to hit a tiny differential element of our detector. This solid angle is hard to think about, but fortunately the Reciprocity Theorem says that we can equivalently think about the element of our detector emitting toward the star at a distance d, and the star has a solid angle

$$\Omega_y = \pi\left(\frac{R_\star}{d}\right)^2$$

Another way of thinking about the solid angle of the star, other than invoking the Reciprocity Theorem, is to imagine it as a circle (projection of a sphere) with an $a$ngular radius $\rho = R_\star/d$. This circle will then have an (angular) area $\Omega_y = \pi\rho^2$.

Putting all this together results in

$$F(d) = \pi B_{\lambda_{\max}}(T_y)\left(\frac{R_\star}{d}\right)^2 \Delta\lambda$$

With the algebraic expression for flux in hand, we can convert to photons per unit time, per area by dividing by the energy of a photon at $\lambda_{\max}$, expressed as $E_\gamma = hc/\lambda_{\max}$:

$$F_{\rm phot}(d) = \pi \frac{B_{\lambda_{\rm max}}(T_y)}{E_\gamma} \left(\frac{R_\star}{d}\right)^2 \Delta\lambda = \boxed{0.02 \text{ photons s}^{-1} \text{ cm}^{-2}}$$

For a 1-meter telescope, and a 15-minute exposure, this corresponds to about 10^5 photons, which is a reasonable signal. Assuming the entire detector system has a 10% efficiency of detecting photons that enter the aperture, this is still a reasonable signal, corresponding to a signal-to-noise ratio of about 100.

(c) Assume the star has a radius equal to that of the Sun, $R_\star = R_\odot$. How many photons arrive from the Sun-like star ($T \approx 5800$ K) in a 1 μm-wide wavelength interval centered at $\lambda_{\rm max}$ of the Y dwarf? Consider what wavelength regime this corresponds to, and what approximation you can use to simplify your calculation.

We can use the same formula for photon flux, $F_{\rm phot}(d)$, as in the previous section, but use $T_\star = 5800$ K

$$\boxed{F_{\rm phot}(d) = 880 \text{ photons s}^{-1} \text{ cm}^{-2}}$$

(d) What is the flux ratio of the Y dwarf to the star near $\lambda_{\rm max}$? This should illustrate why it's so difficult to detect substellar companions around Sun-like stars.

The calculations above correspond to a flux ratio of $\boxed{4.4 \times 10^4}$. This means that for every 44,000 photons arriving from the star, only 1 photon is from the Y dwarf!