Math 137 - Problem Set 1
Due Wednesday, Feb 5

All rings are commutative, and k is an algebraically closed field.

1. Let $R = k[x_1, \ldots, x_n]$. Verify the following statements from class.

   (a) Let $\{I_\alpha\}$ be a collection of ideals in R. Show that $\bigcap_\alpha V(I_\alpha) = V(\cup_\alpha I_\alpha)$.

   (b) Let $I, J \subset R$ be ideals. Show that $V(IJ) = V(I) \cup V(J)$.

2. Let $J \subset k[x_1, \ldots, x_n]$ be an ideal and X and Y algebraic sets. Verify the following statements.

   (a) $V(I(V(J))) = V(J)$.

   (b) $I(V(I(X))) = I(X)$.

   (c) $X = Y$ if and only if $I(X) = I(Y)$.

3. Let $f \in k[x, y]$ be a polynomial of degree $n > 0$, and $C = V(f)$. Let L be a line in $\mathbb{A}_k^2$ such that L is not contained in C. Show that $L \cap C$ is a finite set of no more than n points.

4. Determine (and prove) whether each of the following is an algebraic set.

   (a) $\{(t, t^2, t^3) \in \mathbb{A}_k^3 \mid t \in k\}$.

   (b) The set of $m \times n$ matrices over k with rank $\leq r$. respectively.

   (c) The closed unit ball $\{P \in \mathbb{A}_{\mathbb{C}}^n \mid \|P\| \leq 1\}$.

   (d) The set $U \times V \in \mathbb{A}^{m+n}$, where U and V are algebraic sets in $\mathbb{A}^m$ and $\mathbb{A}^n$, respectively.

5. Let R be a ring. Show that the following statements are equivalent. (Hint: You will need to use Zorn's Lemma.)

   (i) R is Noetherian.

   (ii) Every strictly increasing sequence of ideals $I_1 \subsetneq I_2 \subsetneq \ldots$ is finite.

   (iii) Every set of ideals $\mathcal{J}$ of R contains an element I such that no other element of $\mathcal{J}$ contains I.