Math 137 - Problem Set 4
Due Wednesday, Feb 26

All rings are commutative, and k is an algebraically closed field.

1. Verify the following claims/assumptions from class.

   (a) Let R be a reduced finitely generated k-algebra (reduced means (0) is radical). Then R is isomorphic to the coordinate ring of some algebraic set.

   (b) Let $\phi, \psi : X \to Y$ be morphisms of algebraic sets. If $\phi^* = \psi^*$, then $\phi = \psi$.

2. Let $\phi : \mathbb{A}^2 \to \mathbb{A}^2$ be the regular map given by $\phi(x, y) = (x, xy)$.

   (a) Show that the image of ϕ is neither open nor closed.

   (b) What is the closure of the image of ϕ?

3. Let $V = V(x^3 - y^2) \subset \mathbb{A}^2$. Consider the morphism $\phi : \mathbb{A}^1 \to V$ given by $\phi(t) = (t^2, t^3)$.

   (a) Show that ϕ is a bijection but not an isomorphism.

   (b) Show that V is not isomorphic to $\mathbb{A}^1$. (Note that this is not implied by (a).)

4. Let $V = (x^2 - y, x^3 - z) \subset \mathbb{A}^3$. Define $\phi : \mathbb{A}^1 \to V$ by $\phi(t) = (t, t^2, t^3)$. Show that ϕ is an isomorphism.

5. For the morphisms defined in the previous three problems, explicitly describe the induced map of coordinate rings.

6. Let $\phi : X \to Y$ be a morphism of algebraic sets. Show that if X is irreducible, the Zariski closure of $\phi(X)$ is irreducible.

7. Let
$$V = \left\{ (x, y, z, w) \in \mathbb{A}^4 \mid \operatorname{rank} \begin{pmatrix} y & z & w \\ x & y & z \end{pmatrix} \leq 1 \right\}.$$
   Show that V is an irreducible variety. *(Hint: Can you find a map from a certain 2-dimensional irreducible variety that has V as its image?)*

Bonus question (Extra credit)

8. Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be algebraic sets. Show that the coordinate ring of $X \times Y \subset \mathbb{A}^{m+n}$ is isomorphic to $\Gamma(X) \otimes_k \Gamma(Y)$.