Math 137 - Problem Set 7
Due Friday, Mar 27

All rings are commutative, and k is an algebraically closed field.

1. Suppose $P = (0, 0)$ is a point of multiplicity two on an affine plane curve f, and suppose f has only one unique tangent L at P.

   (a) Show that $I_P(f, L) \geq 3$. The curve f is said to have a ***cusp*** at P if equality holds.

   (b) Suppose $L = y$. Show that P is a cusp if and only if $f_{xxx}(P) \neq 0$, where f_{xxx} is the third partial derivative of f with respect to x.

   (c) Show that if P is a cusp on f, then f has only one component passing through P.

2. Let $f \in k[x_1, \ldots, x_{n+1}]$. Write $f = \sum_i f_i$, where f_i is a form of degree i. Let $P \in \mathbb{P}^n$ and suppose $f(a_1, \ldots, a_{n+1}) = 0$ for every choice of homogeneous coordinates $[a_1 : \ldots : a_{n+1}]$ for P. Show that for all i, $f_i(a_1, \ldots, a_{n+1}) = 0$ for every choice of homogeneous coordinates for P.

You might want to wait until after class on Monday 3/23 to work on the remaining problems.

3. Let $V \subset \mathbb{P}^n$ be an algebraic set and I a homogeneous ideal. Verify the following claims from class:

   (a) If $V \neq \emptyset$, then $I_a(C(V)) = I_p(V)$.

   (b) if $V_p(I) \neq \emptyset$, then $C(V_p(I)) = V_a(I)$.

4. Let $I \subset k[x_1, \ldots, x_{n+1}]$ be a homogeneous ideal.

   (a) Show that I is prime if and only if for any forms (i.e. homogeneous polynomials) $F, G \in k[x_1, \ldots, x_{n+1}]$, such that $FG \in I$, we have $F \in I$ or $G \in I$. (One direction should be obvious.)

   (b) Show $\sqrt{I}$ is homogeneous.

5. Let $S = k[x_1, \ldots, x_{n+1}]$, $I \subset S$ a homogeneous ideal, and $R = S/I$.

   (a) Let $d \geq 0$ and $S_d \subset S$ the set of forms of degree d. Show that S_d is a k-vector space and calculate its dimension.

   (b) Let R_d be the image of S_d in the quotient. Show R_d is also a k-vector space of finite dimension.

6. Let $S = k[x, y, z]$, and $f \in S$ an irreducible form of degree n. Let $C = V(f) \subset \mathbb{P}^2$, and $R = S/(f)$.

(a) Show that there is a short exact sequence

$$0 \to S \xrightarrow{\cdot f} S \to R \to 0,$$

where the map from S to S is multiplication by f, and the second map is the natural quotient. (i.e. this amounts to showing that the first map is injective, and the second map is the quotient of the first.)

(b) For each d, calculate $\dim_k(R_d)$. Keep in mind that your answer may depend on whether $d > n$, $d = n$, or $d < n$.